Mathematics

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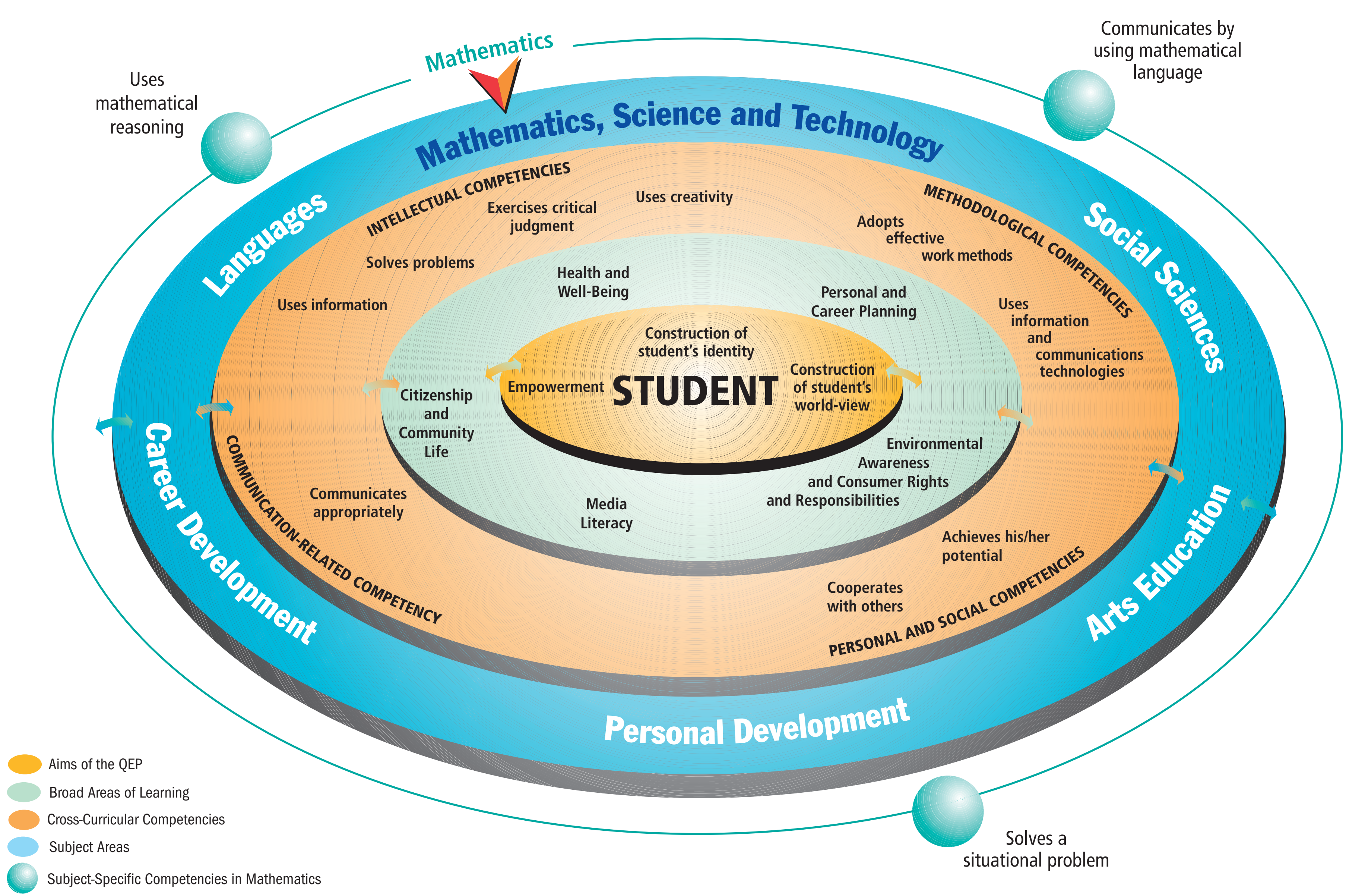
Making Connections: Mathematics and the Other Dimensions of the Québec Education Program (QEP)



Introduction to the Mathematics Program

Mathematics is the tool specially suited for dealing with abstract concepts of any kind and there is no limit to its powers in this field.
Paul Adrien Maurice Dirac

Mathematics is a science and a universal language that plays an important role in a person's intellectual, social and cultural development. It provides keys for understanding reality and makes it possible to derive models and perform various operations on them. It gives students the tools they need to adjust to a changing world, to make the most of their intuition, creativity and critical judgment, and to make decisions. In so doing, mathematics helps them to construct their identity and world-view and to become empowered. It prepares them to act as thoughtful and responsible citizens in society.

Mathematics plays an important role in a person's intellectual, social and cultural development.

Mathematics is an intrinsic part of daily life and social progress, and is used in virtually every field (e.g. the media, the arts, business administration, biology, engineering, design, sports). The variety of meaningful situations that can be examined by means of mathematics or from which mathematics derives its structures show the extent to which it is related to a multitude of everyday activities. Its many different applications cannot, however, be appreciated or understood without acquiring some basic knowledge of its various branches. Arithmetic and algebra can be used to interpret quantities and their relationships, geometry can be used to interpret space and shapes, and statistics and probability can be used to interpret random phenomena.

Designed as a continuation of the Secondary Cycle One program, this mathematics program is aimed at developing the following closely linked competencies that have the same relative importance:

– Solves a situational problem
– Uses mathematical reasoning
– Communicates by using mathematical language

Although these competencies are, for all practical purposes, part and parcel of mathematical thinking, they are distinguished by the fact that they focus on different facets of that thinking. This distinction should make it easier to organize the instructional process without compartmentalizing the study of the elements specific to each competency. In addition, while mathematics, as a language and an abstraction tool, requires that the relationships between objects or elements of situations be examined in the abstract, secondary-level mathematics education is more effective when it involves real-world objects or situations.

The mathematics program is aimed at developing closely linked competencies that have the same relative importance.

The solving of situational problems is the essence of mathematical activity and, in this program, is examined from two perspectives. On the one hand, it is viewed as a process, which is embodied in the competency *Solves a situational problem.* Problem solving is of particular importance because conceptualizing mathematical objects involves applying logical reasoning to situational problems. On the other hand, problem solving is also an instructional tool that can be used in most mathematical learning processes.

The competency *Uses mathematical reasoning* is the cornerstone of all mathematical activity. Mathematical reasoning involves much more than just idea-structuring processes or the oral or written presentation of a result. Students who use mathematical reasoning must organize their thinking by attempting to understand a body of knowledge and the interrelationships between these items of knowledge. By analyzing and working with all types of situations, students learn to conjecture, that is, to presume that a statement is true and to try to validate it on the basis of an argument or proof. This is how they develop their ability to convince themselves and others. The main types of reasoning used are analogical, inductive and deductive. Proof by exhaustion or proof by contradiction as well as refutation by means of counterexamples are also used in various types of situations.

Since language is the vehicle for thought, the competency *Communicates by using mathematical language* is essential to understanding and conceptualizing mathematical objects. It is therefore indispensable for developing and using the other two competencies. Three objectives are pursued: to become familiar with and to consolidate the elements of mathematical language (e.g. vocabulary and the different meanings of a known word as well as the different registers of semiotic representation);[1] to interpret or formulate a message that involves explaining a procedure or a line of reasoning; and to meet certain requirements of communication. Thus, students will have to know how to draw up a communication plan, to take their audience into account in their choice of mathematical tools, to choose a speaking or writing style depending on whether they wish to provide information, a justification or a proof, and to be open to different viewpoints.

Mathematics is an integral part of our everyday lives and cultural heritage. The relationships between mathematics and other fields of knowledge as well as between the different branches of mathematics are a source of enrichment and permit a better understanding of the scope of this subject. It is therefore crucial that students develop the mathematical literacy that enables them to understand the different roles played by mathematics, to participate in activities that involve using mathematics, to trace the evolution of mathematics over time, to discover how different needs have been met through mathematics and to learn how researchers with a passion for mathematics have contributed to its development.

Diversified Paths

In Secondary Cycle Two, the mathematics program offers three different options designed to meet students' needs. They are the *Cultural, Social and Technical* option, the *Technical and Scientific* option and the *Science option.*

In the first year of Cycle Two, students complete their basic education and choose their option for the following year. This choice must be as consistent as possible with their aspirations, interests and aptitudes. To help them in this regard, the teacher suggests mathematical activities that will familiarize them with the characteristics of each option (e.g. aims, mathematical content, tasks and projects). In specific cases, students whose aspirations or interests change will, under certain conditions, be able to choose another option at the beginning of the last year of the cycle.

In the first year of Cycle Two, students complete their basic education and choose their option for the following year.

Each of the three options involves complex and meaningful learning situations, in concrete or abstract contexts. The use of technology—which is part and parcel of everyone's daily life—is considered a valuable tool in dealing with various situations. By making it possible to explore, simulate and represent a large number and variety of complex situations, technology fosters both the emergence and understanding of mathematical concepts and processes. It enables students to work more effectively in carrying out the tasks assigned to them.

In the last year of Cycle Two, students are required to complete an independent assignment related to the option they have chosen. This assignment has a four-fold purpose: to help students develop a positive attitude toward mathematics, to help them cultivate an appreciation for its cultural and work-related significance, to stimulate their curiosity and to give them opportunities to use their mathematical competencies.

1. A *register of semiotic representation* is a system of "recognizable traces" that consists of rules of conformity, transformation and conversion. The different registers of semiotic representation in mathematics are linguistic, symbolic, iconic and graphical. See Appendix D.

A Description of the Different Options

The different options prepare students to enter certain trades, professions or technical occupations and to become better integrated into society. All of these options allow students to enroll in a preuniversity program. The development of mathematical literacy, the role of active citizens and the requirements of different fields of employment are addressed in all three options. Although all three options foster exploration, experimentation and simulation, each one has its own course of study and aims.

Cultural, Social and Technical Option

The *Cultural, Social and Technical* option is intended for students who like to design objects and activities, develop projects or participate in making them or carrying them out. It stimulates students' interest in social causes and helps them develop their sense of initiative. It involves a greater use of statistics and discrete mathematics,[2] and emphasizes situations that students will encounter in their personal and professional lives. It brings together aspects of mathematics that will help students become autonomous citizens who are active and thoughtful members of society. The learning content for this option allows students to build on their knowledge of basic mathematics. Specifically, it prepares them for studies in the arts, communications, the humanities and the social sciences.

Technical and Scientific Option

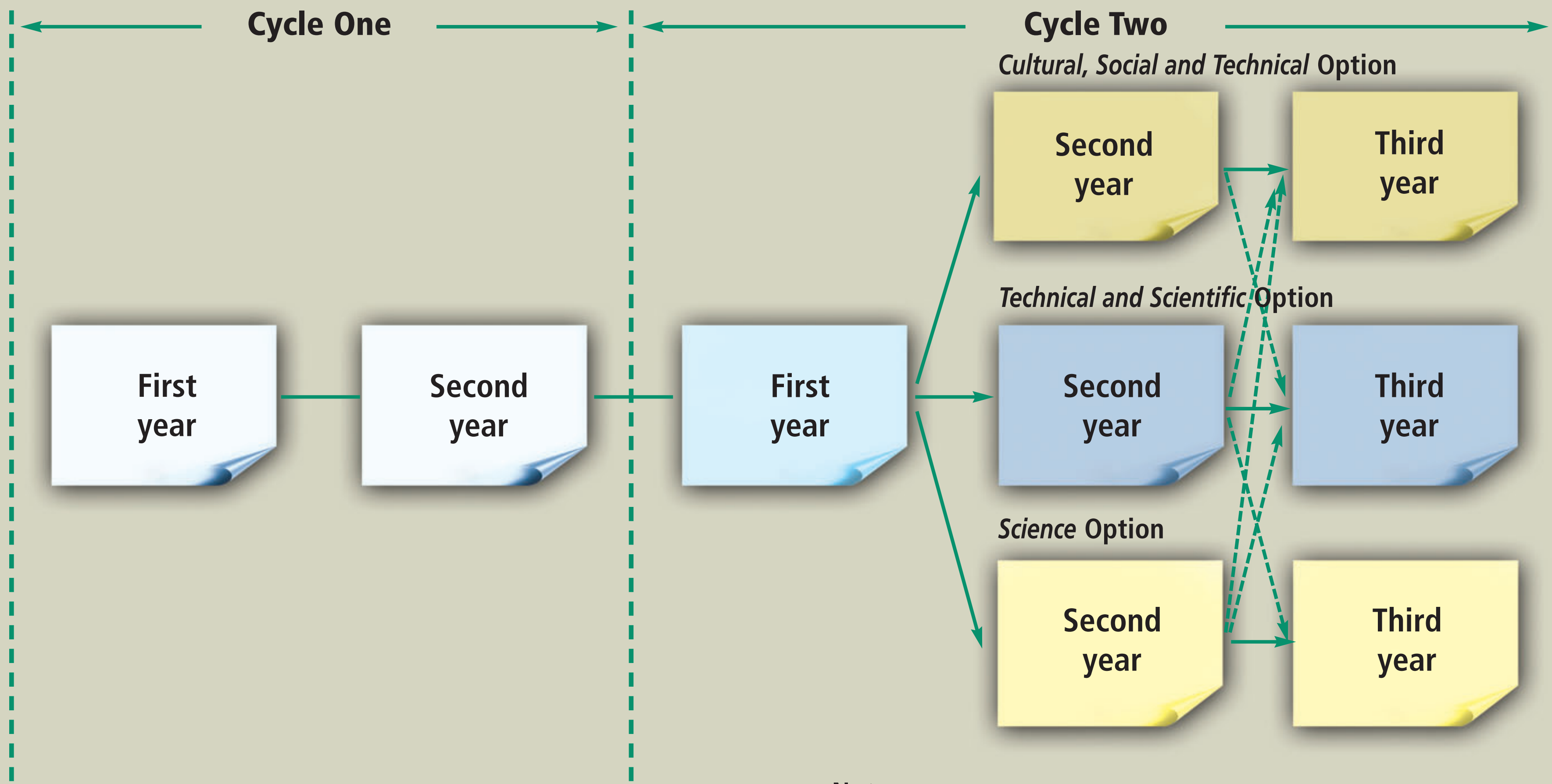
The *Technical and Scientific* option is intended for students who wish to explore situations that sometimes involve both manual and intellectual work. The emphasis is on case studies as well as the development of students' ability to identify errors and anomalies in processes or solutions, with a view to defining the problem and taking appropriate corrective action. It also requires students to identify the mathematical concepts and processes associated with the design, operation or use of certain technical instruments. This option encourages the exploration of different areas of study, but it is especially designed to equip students to work effectively in technical fields related to nutrition, biology, physics, business administration, the fine arts and graphic arts.

Science Option

The *Science* option is intended for students who seek to understand the origin of different phenomena and how they work, as well as to explain them and make decisions that pertain to them. Students learn to develop formal proofs in situations where there is always a need to confirm a truth. By focusing on the properties of mathematical objects, this option places greater emphasis on students' capacity for abstract thinking in that they are required to perform more complex algebraic operations. The emphasis is on finding, developing and analyzing models within the context of experiments mainly related to different scientific fields. Students who choose this option develop strategies and acquire an academic background that specifically enables them to pursue their studies in the hard sciences or to perhaps eventually specialize in research.

2. Discrete mathematics is a branch of mathematics that focuses mainly on finite situations involving finite sets and countable objects.

SECONDARY-LEVEL MATHEMATICS: THE STUDENT'S PATH



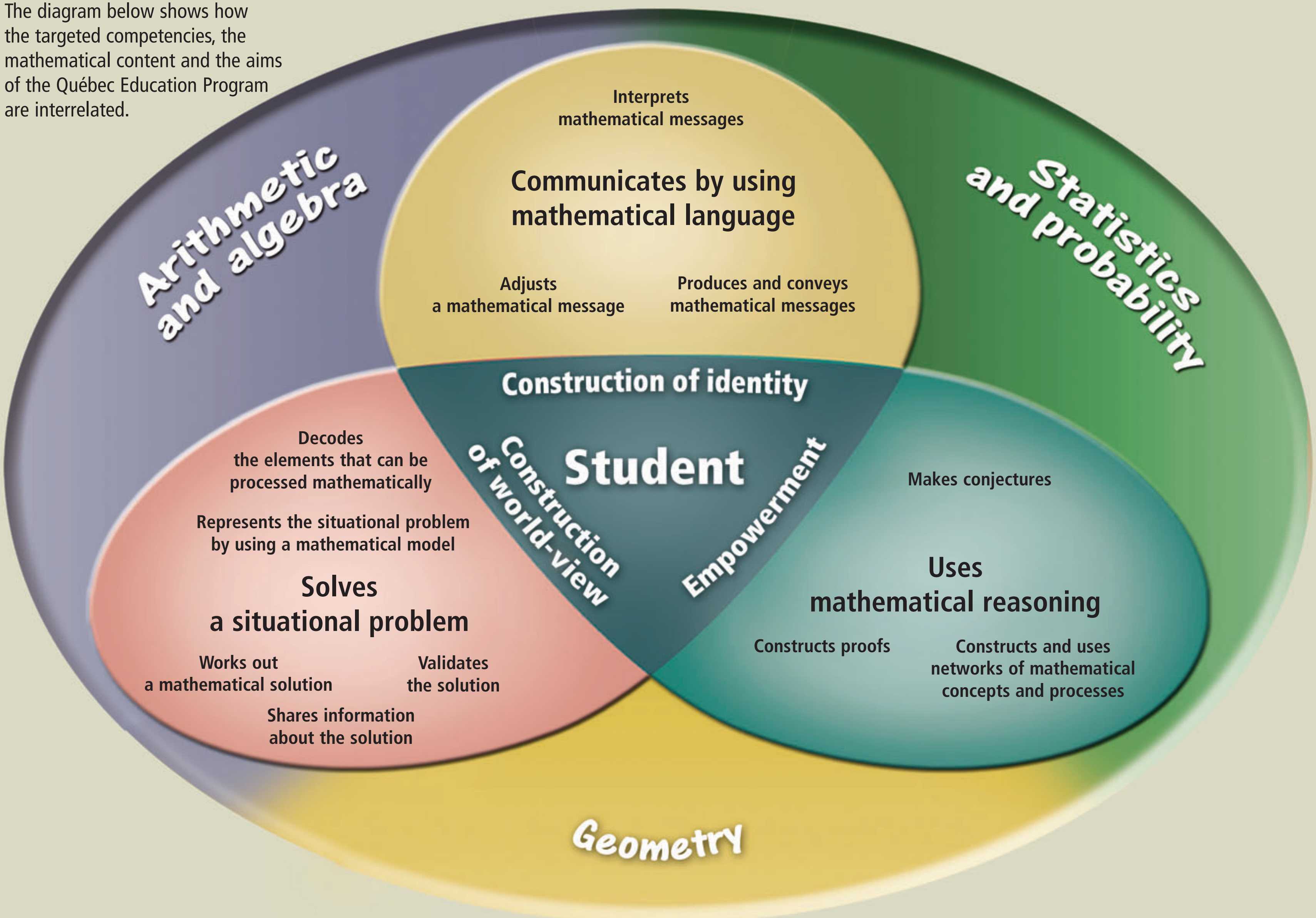
Note:

At the end of the second year of Cycle Two, students may switch from one option to another, but only in certain specific cases.

In addition, switching from the *Cultural, Social and Technical* option to the *Science* option may require a compensatory measure, to be determined by the school, in order to make up for the differences in the instructional time allotted to each option.

CONTRIBUTION OF THE MATHEMATICS PROGRAM TO THE STUDENT'S EDUCATION

The diagram below shows how the targeted competencies, the mathematical content and the aims of the Québec Education Program are interrelated.



Making Connections: Mathematics and the Other Dimensions of the Québec Education Program

"Without the help of mathematics," the wise man continued, "the arts could not advance and all the sciences would perish."
Júlio César de Mello e Souza, alias Malba Tahan

Replete with everyday applications, mathematics can be used to enhance instruction as it pertains to the components of the Québec Education Program, namely the broad areas of learning, the subject-specific competencies and the cross-curricular competencies. In turn, mathematics education is enriched by the many connections that can be made between mathematics and these various components.

The broad areas of learning deal with major contemporary issues. Through their specific approaches to reality, the various subjects illuminate particular aspects of these issues and thus contribute to the development of a broader world-view.

Connections With the Broad Areas of Learning

The educational aims and the focuses of development of the broad areas of learning provide the context for developing learning situations that promote the acquisition of subject-specific and cross-curricular competencies. They make it possible to relate academic learning to the students' concerns. Because mathematics in all its forms is such an integral part of everyday life, connections can be made between mathematics education and each of the broad areas of learning.

Health and Well-Being

When encouraged to reflect on the process they have undertaken to acquire healthy lifestyle habits, students can draw on their knowledge of modelling and statistical processing to predict the impact of certain decisions when examining issues relating to proper nutrition, sexuality, behaviours or living habits. When expressing their opinions and needs in this respect, they may also be required to use the competency *Communicates by using mathematical language.*

For example, in order to determine why trans fats are harmful to health and how to remedy this problem, students collect data, then organize and interpret it. They then express their viewpoint and formulate recommendations with respect to nutrition. For a number of foods, they may establish certain relationships between cholesterol and calories, between fat and protein, or between the amount of fat and "bad" or "good" cholesterol.

Career Planning and Entrepreneurship

Starting in the first year of Cycle Two, students are encouraged to reflect on their preferences, areas of interests and aptitudes, since the option they select at the end of this first year may influence their career choices. When they are given situational problems dealing with the labour market, they are required to reflect on their talents, qualities and personal and career aspirations. Such situations also give them an opportunity to increase their knowledge of the world of work, social roles and the requirements of different trades, occupations or professions.

To develop their mathematical competencies, students must also master various affective, cognitive, metacognitive and resource management strategies,[3] which can be used in any type of situation, in order to carry out projects and see them through to completion. Students must also learn about group work, which prepares them to enter the labour market, where the ability to cooperate and work in a team has become an invaluable asset.

Lastly, it should be noted that mathematical knowledge provides students with the tools they need to explore various career paths. For example, a knowledge of geometry may give them a taste for occupations that involve

3. For example, management of time, the environment, or human and material resources.

the representation or construction of objects. Their ability to create models and make predictions, two skills developed in all branches of mathematics, will help them explore spheres of activity that require qualities such as a sense of pride in work well done and a sense of initiative. Statistics and probability will provide them with other opportunities to discover social and scientific fields.

Environmental Awareness and Consumer Rights and Responsibilities

Mathematics provides students with a number of resources for gaining some perspective on their relationship to the environment and consumer society. By developing their ability to generalize and to establish various relationships between variables (e.g. cause and effect, dependency, chance), students will have a better grasp of the interdependence between the environment and human activity. The work they do in learning situations that involve personal finances and business plans equips them to make informed choices when it comes to purchasing goods and balancing their budget. Such situations also help them to understand the economic repercussions of their actions and choices. Lastly, when they use a line of reasoning to explain or validate a behaviour or phenomenon, they must consider certain social, ethical and economic aspects of consumption or the environment.

For example, if they are asked to conduct a study on factors that may influence the purchase of a car (in order to come up with a catchy advertising concept, compare leasing and purchasing options or become aware of the importance of the quality-price ratio when choosing a vehicle), they will use their spatial sense as well as their data processing, modelling and measurement skills. Comparing the depreciation values of certain consumer products, evaluating the efficiency of a water purification or waste management system, and analyzing the impact of polluting emissions are other examples of situations in which students use their mathematical knowledge to exercise critical judgment, make decisions or formulate recommendations with respect to consumption or the environment.

Media Literacy

Mathematical competencies can contribute to the development of critical, ethical and aesthetic judgment with respect to the media. When they analyze and produce messages in which they use materials and codes specific to media communication, students recognize the different representations of mathematical objects, distinguish among them and gauge their appropriateness. They make sure that the mathematical information in the message is plausible. They use their number sense, their ability to analyze data and their communication skills to determine the intention of those conveying the message and the sources of bias that can influence their judgment. They use their capacity for mathematical reasoning to differentiate between fact and opinion.

Their spatial sense and knowledge of shapes, geometric figures[4] and proportions can also help them develop criteria for assessing media representations in terms of image and movement and for evaluating the aesthetic quality of a message.

Citizenship and Community Life

Some of the activities students carry out in mathematics class introduce them to the requirements of the democratic process. They must often make choices and decisions that require them to consider different viewpoints and opinions. In such cases, analogical reasoning, which mathematics helps to develop, is very useful in establishing similarities between different viewpoints and facilitating consensus building within a group. When they validate a conjecture or solve a situational problem, students use persuasive arguments and justifications and develop compromise solutions in order to complete tasks in accordance with existing agreements and commitments. Work that calls for cooperation with their peers also teaches students to observe certain principles, rules and strategies related to teamwork.

Mathematics contributes to this broad area of learning in many different ways. For example, students can use their knowledge of optimization to carry out any type of planning that involves taking into account work distribution, time management, distance or cost determination and the number of people concerned. When they analyze situations that involve probability, they can examine the diversity of beliefs held by the members of a society. To sum up, mathematics makes students aware of the need to question their own perceptions, to consider several possibilities and to analyze a situation objectively.

4. At the secondary level, a geometric figure refers to the representation of a geometric object with 0, 1, 2 or 3 dimensions.

Mathematical competencies contribute to the development of cross-curricular competencies to varying degrees. Conversely, cross-curricular competencies are important resources in helping students use mathematical competencies.

Connections With the Cross-Curricular Competencies

The cross-curricular competencies are not developed in a vacuum; they are rooted in specific learning contexts, which are usually related to the subjects.

Intellectual Competencies

Faced with a situation that can be processed mathematically, students use information when they collect data, identify the relevant elements of the situation, summarize information, synthesize their knowledge, or when they interpret, produce or validate a message. In so doing, they make connections between what they already know and new information, distinguish between essential and secondary information and evaluate the validity of information according to certain criteria.

The competency *Solves a situational problem,* which has long been associated with mathematics, overlaps with the cross-curricular competency *Solves problems.* The ability to use these competencies leads to similar results. They share several strategic elements and reflect a similar approach to asking questions and examining situations. They differ in terms of their specific focus. By developing the ability to confront problems, students will become citizens who are able to deal with novelty, define issues and arrive at possible solutions that take into account a number of factors, constraints, effects or results.

Every time students use mathematics to examine any given information, they are in a position to exercise their critical judgment. They analyze a situation, formulate a conjecture, work out a proof, justify a solution, a choice or a decision, assess the appropriateness of the semiotic representation, and so on. They are able to validate their conjectures by using definitions, theorems and postulates that enable them to construct a solid argument.

Students also use their creativity when they apply their mathematical competencies. This is the case when, in trying to solve a situational problem, they come up with several solutions, explore various models,[5] avenues and strategies, listen to their intuition, accept risks and unknowns, play with ideas, try out new approaches, explore new strategies and express their ideas in new ways.

Methodological Competencies

Being able to use mathematical competencies implies the acquisition of effective work methods that combine rigour and flexibility. Whether solving a situational problem or using mathematical reasoning, students must structure their thinking and organize their approach. When presenting their approach and explaining their reasoning, they must use appropriate registers of semiotic representation and observe the rules associated with the different ways they may choose to present their work.

By mastering mathematical processes and strategies, students can make connections between the work methods they must develop in mathematics and certain aspects of this cross-curricular competency. It is also important that they become aware of their personal learning style so that they adopt work methods that correspond to their needs and way of doing things.

Mathematical competencies play an important part in developing the ability to use information and communications technologies. Historically, mathematics has contributed to the development of these technologies on both a theoretical and technical level. Conversely, these technologies make it easier to conduct and advance mathematical research. They help students carry out studies, create simulations and models, formulate conjectures, manipulate large amounts of data and a multitude of geometric figures, and represent them in different forms.

Personal and Social Competencies

The development of mathematical competencies involves placing students in situations where they must deal with novelty and show autonomy and self-confidence, while recognizing the influence of others. Each time they express themselves or make choices or decisions, they interact with their peers and teacher. Whether they are attempting to inform, explain or convince, they must compare their perceptions with those of others and respond to the information they receive in the process. In so doing, they take their place among their peers, become engaged in what they are doing, assess the quality and appropriateness of their actions, recognize the impact of these

5. In mathematics, a model is a concrete, conceptual or operational representation of a fragment or aspect of reality.

actions on their successes and difficulties, evaluate their progress and persevere in an effort to achieve their goals. All of this will help them fulfill their potential.

The various mathematical activities that call for cooperation give students the opportunity to exchange points of view, share solutions, explain their ideas and make an argument to defend their opinions, justify their choices and actions, or convince others of the efficiency of a solution, thereby developing their ability to cooperate with others. Among other things, they may be required to compromise in order to achieve a group objective, to plan and carry out a project with others, to resolve conflicts, to evaluate their contribution and that of their peers and to adapt their behaviour to other people and the task at hand.

Communication-Related Competency

When students communicate by using mathematical language, they interpret, produce, convey or adjust their message. The development of this mathematical competency is closely linked to the competency *Communicates appropriately.* Students are required to decode a message, define their intentions, express their viewpoint, compare their ideas with those of others, formulate conjectures, explain a line of reasoning, present their results, discuss their solution with their peers, adjust their communication style according to the reactions of their audience, and write proofs, summaries, syntheses, and so on. In all circumstances, they are required to express themselves in appropriate mathematical language and to expand their knowledge of different forms of representation.

Reality can rarely be understood through the rigid logic of a single subject; rather, it is by bringing together several fields of knowledge that we are able to grasp its many facets.

Connections With the Other Subject Areas

Making connections between mathematics and other subjects enriches and contextualizes the learning situations in which students will be developing their competencies. The examples below, grouped by subject area, reflect the many different connections that can be made between mathematical knowledge and some of the other subjects.

Languages

The mastery of language and the use of certain strategies related to the study of language[6] help students develop and use mathematical competencies. These strategies make it possible to understand the elements of a situational problem, to work out a solution and to communicate it. Furthermore, language is necessary for forming networks of mathematical concepts and processes and for formulating and validating conjectures. Lastly, the ability to switch from one register of semiotic representation to another is dependent on students' language skills and, furthermore, these skills allow students to enhance the visual elements involved in organizing graphics and text.

Learning the language of instruction or a second language and learning mathematics are similar in a number of ways. The study of both language and mathematics involves situations that require students to communicate different types of messages. Language education and mathematics education require the use of cognitive and metacognitive strategies as well as resource management strategies, mainly as they pertain to the interpretation of information, planning, the organization of ideas and the definition of a procedure. They help students develop a concern for using exact language, the ability to present an argument or an analysis from different viewpoints as well as the capacity for reasoning, which includes analogical, inductive and deductive reasoning. Both use a heuristic approach in which the formulation of conjectures and the observation of patterns make it possible to generalize, especially when attempting to determine a rule. In short, they help people learn to think on an abstract level, to examine situations critically and to express themselves logically.

Science and Technology

Mathematics is closely linked to science and technology, which deal with a variety of problems that call for the development of mathematical models. Conversely, these models help to promote an understanding of scientific phenomena and the development of technology. Students use their mathematical reasoning in processing observed or collected data of a scientific or technological nature. Solving a situational problem and seeking

6. For example, reading, parsing, writing, listening, speaking, editing and correcting skills.

answers or solutions to scientific or technological problems involve similar processes (e.g. decoding situations, creating models, and developing and validating solutions). Lastly, students are required to use the competency *Communicates by using mathematical language* when they represent, manipulate and interpret data.

Social Sciences

In History and Citizenship Education, when students examine social phenomena from a historical perspective and interpret them using the historical method, they use mathematical reasoning and their ability to communicate by means of mathematical language. For instance, they use their number sense, their spatial sense, proportional reasoning and statistical tools in order to analyze situations and formulate and support their opinions. The use of cultural references with a historical connotation may help students make connections between the two subjects. For example, in situating mathematical concepts and processes in the period in which they arose and identifying the needs that they addressed, students become aware of the social phenomena associated with different eras and come to understand the human dimension involved in the development of mathematical knowledge.

Arts Education

Students may be required to use their mathematical knowledge when they create or appreciate personal and media images or dramatic, musical and choreographic works. Since the creative dynamic and the solving of situational problems are both based on creativity and intuition, they involve similar approaches and require a sense of organization in the implementation process. Students use their ability to communicate in mathematical language when naming or representing figures or transformations. They use their spatial sense to organize personal and media images and to create dramatic, musical or choreographic works.

Personal Development

In Physical Education and Health, the development of the competency *Adopts a healthy, active lifestyle* may require students to use proportional reasoning, their number sense and their ability to process data in order to analyze situations related to their food consumption, their state of health or athletic performance, for example.

The notions and concepts associated with argumentative thought that are studied in the Ethics and Religious Culture program can be directly applied to the use of mathematical reasoning. In developing the dialogue-related competency, students must distinguish between the various argumentative practices that can hinder dialogue. They identify the different types of reasoning and judgment and are able to recognize a thesis. They use sound arguments to develop their point of view. This process helps them to reason clearly and to support a viewpoint, and overlaps with mathematics, especially in the development of proofs. More generally, reflection, analysis, questioning and justification viewed as a heuristic process in ethics and religious culture can help students organize their reasoning and solve situational problems. Some themes give rise to connections between these two subjects. For example, a topic such as the legalization of drugs can be examined in terms of its economic impact and the health risks it entails. Furthermore, discussions on the definition of happiness can focus on the relative importance of certain factors that may shape this definition. In such activities, students also use their ability to communicate when they share their interpretation of different items of information or produce a report on their procedures and conclusions.

Exploration of Vocational Training, Personal Orientation Project and Integrative Project

There are many opportunities to make connections between mathematics and the subjects that focus on vocational guidance. This can be done by using learning situations related to the broad area of learning *Career Planning and Entrepreneurship* and by taking into account the career concerns associated with each of the three options in the mathematic program, which correspond to different areas of interest and fields of activity. This can also be accomplished by encouraging students to develop their autonomy and by enabling them to discover how the strategies they acquire in exercising their mathematical competencies can be useful to them in other areas. Note also that the integrative project provides each student with a chance to discover the cross-curricular nature of mathematics through his or her own specific project.

CONNECTIONS BETWEEN MATHEMATICS AND OTHER SUBJECTS

The following diagram shows some of the connections that can be made between mathematical knowledge and other subject areas.

– Solves a situational problem
– Uses mathematical reasoning
– Communicates by using mathematical language

ARTS EDUCATION

Music
- Number sense (intervals, ratios, rapports)
- Geometric transformations

Visual Arts
- Proportional reasoning
- Graphs
- Geometric figures and spatial sense: projections, geometric transformations, trigonometry, measurements

Drama
- Proportional reasoning
- Spatial sense: projections
- Graphs: critical path

Dance
- Number sense
- Spatial sense and geometric transformations
- Graphs: critical path

- Cultural references

PERSONAL DEVELOPMENT

Physical Education and Health
- Spatial sense
- Proportional reasoning
- Processing of data: measures of position and central tendency, correlation, probability
- Optimization
- Graphs

Ethics and Religious Culture
- Number and operation sense
- Proportional reasoning
- Processing of data: correlation
- Optimization

- Cultural references

SOCIAL SCIENCES

History and Citizenship Education
- Proportional reasoning
- Processing of data
- Spatial sense: measuring and locating
- Graphs
- Cultural references

LANGUAGE OF INSTRUCTION

- Reading and writing statements and conjectures
- Understanding a situational problem
- Using different registers of representation
- Interpreting and organizing qualitative and quantitative data
- Using connectors and quantifiers
- Explaining, describing, making an argument, justifying and refuting
- Cultural references

SCIENCE AND TECHNOLOGY

- Number sense (scientific notation)
- Proportional and other types of reasoning
- Processing experimental data: dependency relationship, functions, modelling, correlation, manipulation of algebraic expressions
- Geometric figures and measurements (e.g. trigonometry)
- Graphs (e.g. molecular diagrams)
- Vectors and loci
- Cultural references

Pedagogical Context

To learn to use their own intellectual resources, human beings must regularly be made to state and solve problems, make decisions, deal with complex situations, carry out projects or research, and test processes whose outcomes are uncertain. If we want students to construct competencies, we must require them to perform such tasks, not just once in a while, but each week, each day, in all kinds of configurations.

Philippe Perrenoud

Several factors affect the quality of learning and tend to make mathematics class a place where students are encouraged to participate actively in their learning and to use their curiosity, creativity, intellectual abilities, manual dexterity and autonomy. Mathematics class must foster the development of subject-specific competencies, while taking into account individual differences and helping to produce committed and competent individuals capable of exercising their critical judgment in different situations.

A Stimulating Environment and the Practice of Differentiation

As a mathematics specialist, the teacher plays a number of roles by guiding, encouraging and motivating students to understand and construct mathematical concepts and processes. The teacher recognizes students' successes and helps them to take stock of their potential as learners and to therefore build their self-esteem. The teacher also regards himself or herself as a learner and a full-fledged member of the school team, working with other team members on a collegial basis.

To encourage students to show commitment and perseverance, the teacher creates a climate that allows each student to carve out his or her place within the class. The teacher uses different approaches, taking into account each student's needs, interests, prior learning and learning pace. The teacher is also concerned with the students' emotional well-being and encourages them to build relationships based on respect for other people's ideas and learning styles. The teacher also helps students to gradually develop the skills and attitudes they need to work with others, exercise their critical judgment and realize their potential.

Since the teacher's goal is to help students develop competencies, that is, the ability to act effectively in a particular context, he or she must help students to become aware, on the one hand, of the way they construct and draw upon their knowledge in various situations and, on the other, of the possibility of reapplying this knowledge by adapting it to new situations. The ability to transfer knowledge is made possible by metacognition. Thus, the teacher provides students with guidance in constructing mathematical concepts and processes and monitors activities in such a way that students can become aware of what they know, what they are doing and the effects of their actions. Furthermore, the teacher gives students a certain latitude for error, which he or she exploits constructively by teaching students to learn from their mistakes or the obstacles they encounter so that they can transform these into resources that enable them to make progress.

The teacher uses pedagogical differentiation in order to help individual students develop their potential as much as possible.

In addition, the teacher uses pedagogical differentiation in order to help individual students develop their potential as much as possible. This practice leads to a variety of learning situations and pedagogical approaches and is predicated on developing a thorough knowledge of each student. Thus, the teacher will consider his or her students' acquired knowledge and previous experiences and try to establish a relationship with them based on trust. Attentive to their needs, the teacher compares the actual results with the expected results and adjusts to his or her students' responses or lack thereof. To elicit responses from students regardless of the branch of mathematics involved, the teacher asks them to work with situations that require justifications or answers to open-ended questions such as "Why?", "Is this always true?" or "What happens when … ?"

Differentiation can be fostered in the following ways:

– drawing inspiration from students' suggestions when devising learning and evaluation situations to achieve certain educational aims
– giving students the opportunity to choose from among a number of situations that involve using the same concepts and processes but in different contexts
– suggesting learning situations that may be used in different branches of mathematics or that involve different registers of semiotic representation
– varying the organization of the class and pedagogical approaches: individual or group activities, situational problems, interactive lecture-style presentations or exploration workshops
– suggesting different types of tasks and projects to take into account the learning style and pace of individual students (e.g. research, journal-keeping, posters, debates, construction, presentations, reports, use of technology)
– encouraging students to devise situations themselves
– using a variety of evaluation methods and tools (e.g. self-evaluation, observation checklists, portfolios, summaries, presentations)

Situations That Optimize Learning

The three competencies in the mathematics program are interrelated and are developed synergistically in meaningful and complexly situations. A situation is meaningful if it relates to students' concerns, stimulates their curiosity and makes them think. A situation is complex if it brings all the key features of a competency into play, poses an intellectual challenge, gives rise to cognitive conflict, fosters risk-taking and lends itself to more than one approach. A situation that is both complex and meaningful therefore encourages students to be active, to draw on their store of experiences and to enrich it with new mathematical knowledge.

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Learning and evaluation situations are centred on the following fundamental aims of mathematical activity[7]: interpreting reality, generalizing, predicting and making decisions. These aims reflect the major questions that have led human beings to construct culture and knowledge through the ages. They are therefore meaningful and make it possible to use the broad areas of learning and to develop subject-specific and cross-curricular competencies by emphasizing the value of mathematics.

Students become active learners when they take part in activities involving reflection, hands-on tasks and exploration in order to construct knowledge or when they have discussions that allow them to express their opinions, justify their choices, compare results and draw conclusions. These situations require them to use their powers of observation, intuition, creativity, intellect, manual dexterity and listening and speaking skills.

In order to stimulate and engage students' interest, the teacher will suggest situational problems, situations involving applications and situations involving communication. These learning situations could involve exploration, hands-on activities, artistic creativity, and so on.

The teacher will suggest situational problems, situations involving applications and situations involving communication.

The solving of situational problems is a valuable pedagogical tool because it entails a variety of high-quality learning activities. When students learn by solving situational problems, they must explore different possible solutions to overcome the obstacles associated with the problem to be solved. They are required to explore concepts and processes in the different branches of mathematics and to construct, broaden, deepen, apply and integrate their knowledge of these concepts and processes. In this way, students use their creativity and acquire the intellectual skills essential to the development of mathematical thinking and a mathematical approach, while learning various affective, cognitive, metacognitive and resource management strategies. They also become aware of their abilities and learn to respect other people's viewpoints.

Often consisting of open-ended problems[8] and sometimes carried out in the laboratory, exploration activities are educationally valuable in that they provide students with the opportunity to formulate conjectures, carry out simulations, perform experiments, develop arguments and draw conclusions. Artistic and recreational activities may also elicit students' interest by enabling them to work in different ways and to use both their creativity and

7. See Appendix A: Aims of Mathematical Activity.
8. Problems for which there are several conceivable solutions, depending on the approaches, points of view or concepts and processes involved, the conditions under which the problem must be solved, etc.

reasoning ability. Situations that involve communication, such as making presentations, participating in discussions and debates, keeping a journal, or writing a research report, an explanation or an algorithm, are also conducive to the development of the competencies outlined in the program. Lastly, activities that allow students to make connections between subjects or within a given subject are also valuable pedagogical tools for using and developing various types of knowledge and for promoting the transfer of learning. Whether it be with regard to arithmetic, algebra, probability, statistics or geometry, students ask questions, use different types of reasoning and develop networks of concepts and processes.

In order to encourage students to explore different career possibilities and better identify their preferences, areas of interest and specific aptitudes, the teacher will present students with learning situations that involve the labour market. At the beginning of the cycle, all occupational fields may be explored so as to help students choose one of the three options. As of the second year of the cycle, when students have already selected an option, they may explore related occupational fields, while focusing on the broad areas of learning. These explorations may involve in-class simulations or may be carried out in the community. For example, students who choose to explore a specific trade or occupation could draw up a list of the skills needed for that type of work and compare these with the competencies they are developing. They can learn about the instruments used in that trade or occupation and identify the mathematical concepts involved in their design, operation or use. They can also consult a resource person with a view to determining the mathematical concepts and processes needed in that line of work. These explorations provide an opportunity to highlight the role of mathematics in society. Discussing these explorations can also be an enriching and stimulating experience for the whole class, which then functions as a learning community.

Students may work on learning and evaluation situations alone or in a group, in the classroom or outside the school, depending on the pedagogical approaches used and the personal development objectives pursued (e.g. autonomy, cooperation, use of effective work methods). These situations involve contexts that are practical and more or less familiar, real or fictitious, realistic or imaginary, or purely mathematical. These activities may be related to the broad areas of learning, cultural references, elements of the program content or an event that occurred in the classroom, in the school or in society at large. Depending on the objectives pursued, the situations involve complete, superfluous, implicit or unknown information. They may lead to one or more results or, on the contrary, lead nowhere.

Strategies That Foster Learning

The teacher must ensure that students continue to develop their competencies. Gauging students' progress in this regard involves considering a number of parameters in determining the complexity of the learning and evaluation situations with which they are presented:

– students' familiarity with the context
– the scope of the concepts and processes involved
– translations between the different registers of semiotic representation
– the existence of intradisciplinary or interdisciplinary links
– the degree of autonomy required of students

These parameters do not necessarily change in a linear fashion. Alternation between simple and complex, between concrete and abstract and between qualitative and quantitative situations is therefore recommended.

The first four parameters listed above link competency development with the construction of the mathematical edifice and are set out under the headings *Development of the Competency and Program Content.* Furthermore, the students' degree of autonomy, needs and motivation as well as their ability to become aware of what they are learning and to choose appropriate strategies represent tools for developing each competency and cannot evolve without the teacher's support.

The teacher helps students become more autonomous so that they can become the architects of their own learning. The teacher regularly encourages them to question what they are learning and how they are learning it.

The teacher guides students in structuring their approaches and helps them make progress in their learning. He or she encourages students to compare their way of doing things with that of their peers in order to expand their repertoire of strategies.[9] He or she guides them so that they become actively engaged in their quest for

9. See Appendix B: Examples of Strategies for Exercising the Competencies.

knowledge, that they recognize that they do have some ability to control, manage and assess their own work, and that they succeed in completing a task or a project on their own.

Students' motivation, the key element in ensuring their commitment, participation and perseverance, is underpinned by the development of a set of affective, cognitive, metacognitive and resource management strategies. These strategies, which are of a specific nature in mathematics, must be developed to varying degrees throughout the cycle and for each of the competencies. The teacher ensures that these strategies are integrated into the students' learning processes.

The development of any competency also involves mastering a set of cognitive strategies. In order to have students succeed in mathematical activities, the teacher must help them manage their thought processes and regularly give them the opportunity to question what they are learning and how they are learning it. The ability to exercise self-control, control the task at hand and regulate cognitive activities can make all the difference between an insecure and a confident student, between a novice and an expert student, or between a competent student and one who is in the process of becoming competent. The teacher's intervention in this regard is crucial.

A Variety of Resources

No one can deny the importance of a hands-on approach in the construction of mathematical concepts. The frequent use of concrete materials constitutes an important tool for learning mathematics in elementary school and in Secondary Cycle One; these materials remain important in later grades because they can promote or facilitate exploration, or lead to the formulation of conjectures or a flash of intuition.

Depending on the activity, students are encouraged to use different resources. Among other things, they may use manipulatives and tools such as geometric blocks, various objects, graph or dot paper, geometry sets, a calculator or software. Some situations involve becoming familiar with instruments such as stopwatches, odometers, oscillographs, sensors and probes, electric circuits or certain tools used in the fields of health, the arts and construction. If necessary, students consult different sources of information, including those found in the library or on the Internet. Students also call upon the help of other people, starting with their teacher but also including their classmates, and make use of other resources in their school or community (e.g. their family; community stakeholders in the fields of employment, sports or recreation; people responsible for health services, academic and career information and guidance, and extracurricular activities).

Although it cannot replace intellectual activity, technology is an extremely useful tool. It allows students to learn by exploring complex situations, to work with large amounts of data, to use different registers of representation, and to perform otherwise tedious simulations and calculations. Students can therefore devote their energy to meaningful activities, use their mental computation skills by approximating the required value and deepen their understanding of mathematical concepts and processes.

Technology fosters both the emergence and understanding of mathematical concepts and processes. It enables students to work more effectively in carrying out the tasks assigned to them.

Software tools are a good example of technology's contribution to the study of mathematics. These tools make it possible to explore and compare different situations, observe variations and patterns, model phenomena and predict results. Dynamic geometry software is a case in point. It allows students to construct figures based on their definitions and properties, to study and manipulate them more easily and to identify certain properties. The use of a graphing calculator or of spreadsheet and graphic software promotes the development of algebraic thinking when students are required to model situations by constructing formulas, algorithms or graphs or by switching from one of these representations to another. By making it easier to work with large amounts of data and to simulate different possibilities, these tools make it possible to analyze a situation and to generalize it through interpolation or extrapolation, among other things.

Selecting the Right Path: An Informed Choice

In the first year of Cycle Two, students complete their basic education in mathematics and choose the option that best suits them. To help students make this choice, the teacher gets them to identify their interests and aspirations and become aware of their reactions, attitudes and preferences in situations related to the different branches of mathematics and the focus of each path. With regard to the required aptitudes, the teacher considers the extent to which students have developed the competencies as well as certain cognitive and metacognitive aspects of their work.

The Purposes of Evaluation

To support learning

Seen as an integral part of teaching and learning, evaluation is planned when the learning situation is devised. Evaluation carried out during the learning process provides both teacher and students with useful information for adjusting an approach, strategies and interventions. When carried out at the end of a given period, evaluation makes it possible to determine the extent to which students have developed a competency so that the next learning sequence can be planned accordingly.

Seen as an integral part of teaching and learning, evaluation is planned when the learning situation is devised.

If students are to overcome obstacles, it is crucial that they and the teacher develop a relationship founded on mutual support and cooperation. Such a relationship is also essential to the evaluation of learning. By involving students in the evaluation process, they learn to take responsibility for their education and become more autonomous. Certain parameters can be verified by the student with the help of the teacher, while others will be verified by the teacher with the help of the student. Students are responsible for developing the metacognitive, affective and cognitive strategies that support the learning process. Tools such as interviews, portfolios, logbooks or other means of recording information can help students evaluate themselves and manage their own progress. For his or her part, the teacher is responsible for evaluating the development of subject-specific competencies and informing students about their learning progress.

By involving students in the evaluation process, they learn to take responsibility for their education and become more autonomous.

To recognize competencies

The competencies in the mathematics program contribute in equal measure to the student's education. They have virtually the same relative importance and are developed in synergy. They may be used, in whole or in part, within the same learning situation. However, when the individual competencies are being evaluated, with a view to preparing an end-of-cycle or end-of-year report, it is preferable to assess each competency through separate situations.

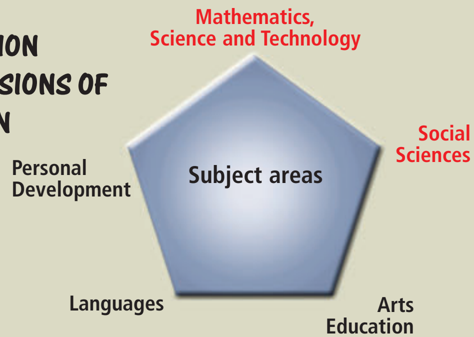
The situational problems used to evaluate the competency *Solves a situational problem* are those that involve a new combination of previously learned concepts and processes. The complexity of a situational problem is dependent in particular on the scope of knowledge and level of abstraction involved, the difficulty of the models to be created and the connections that must be made between the different branches of mathematics.

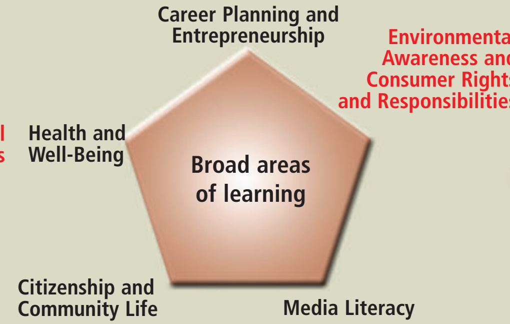
When situations involving applications are used to evaluate the competency *Uses mathematical reasoning,* concepts and processes that have already been learned must be combined in a known way. In addition, these situations require students to explain a line of reasoning when assessing a conjecture formulated by them or by others. These situations are considered to be simple if they deal with one network of concepts and processes. They are regarded as complex if they involve several networks of concepts and processes.

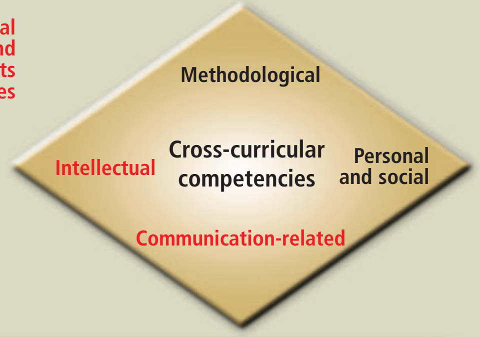
Situations that involve mathematical communication and that serve to evaluate the competency *Communicates by using mathematical language* involve the use of registers of semiotic representation and mathematical concepts or processes with which students are already familiar. This type of evaluation can be carried out orally or in writing. The complexity of the situation with respect to the mathematical knowledge concerned depends largely on the translation from one register of semiotic representation to another.

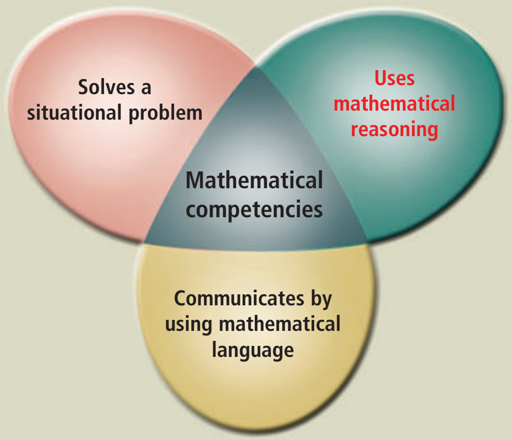
In order to illustrate the characteristics of a pedagogical context conducive to the development of mathematical competencies, the following diagram provides a description of a learning situation and shows how it can be connected to various dimensions of the Québec Education Program.

OVERVIEW OF A SITUATION INCORPORATING DIMENSIONS OF THE QUÉBEC EDUCATION PROGRAM









INTERPRETS REALITY

GENERALIZES

ANTICIPATES

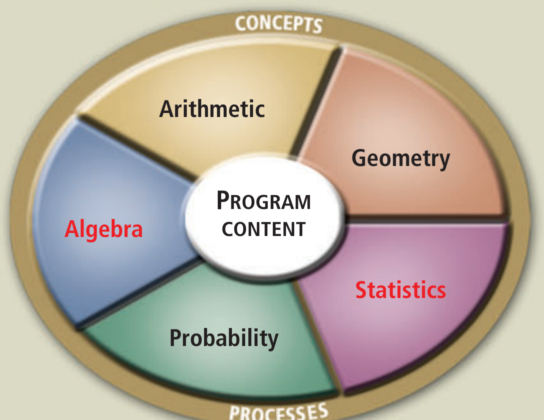
Learning Situation

During an activity on the relative value of incomes, students will become aware of certain social, economic and ethical aspects of consumption. Students will use their ability to reason by mainly combining elements of algebra and statistics.

A study on income is carried out. Income is analyzed according to different factors: age, number of years of education, and value of movable and immovable assets. Each student draws a graph of incomes associated with a representative sample that he or she may or may not have drawn up.

The situation requires students to use their mathematical knowledge and take into account social factors (e.g. popular beliefs, economic status of the region, poverty line) in order to formulate conjectures and determine a functional model for a given population. Deciding on this overall model involves comparing the resulting graphs and using analogical and inductive reasoning. Students are therefore encouraged to make connections between the algebraic and statistical work required in this learning situation. They can also compare their model with that of Statistics Canada or other sources. Each student writes up a report on the activity.

MAKES DECISIONS



In red: Elements of the Québec Education Program that can be connected to the learning situation

COMPETENCY 1 Solves a situational problem

An expert problem solver must be endowed with two incompatible qualities, a restless imagination and a patient pertinacity.
Howard W. Eves

Focus of the Competency

What characterizes a situational problem? A mathematics problem is considered a situational problem if it meets one of the following conditions:

- The situation has not previously been presented in the learning process.
- Finding a satisfactory solution involves using a new combination of rules or principles that the student may or may not have previously learned.
- The solution, or the way in which it is to be presented, has not been encountered before.

Solving situational problems is one of the cornerstones of mathematical activity, and it involves using a heuristic approach (i.e. an approach centred on exploration and discovery). This approach makes it possible to construct mathematical objects, to give them meaning, to draw upon what one already knows, to develop strategies[10] and to adopt a range of attitudes associated with self-confidence and autonomy. *Solves a situational problem* is therefore a complex competency and in using it, students call upon their reasoning skills and develop creative intuition. This competency equips students to deal with novelty and to meet challenges that are within their reach.

Solving a situational problem is a dynamic process that involves constantly going back and forth among the problem-solving stages and calls for the ability to anticipate, use discernment and exercise critical judgment. In developing and using this competency, students must decode the elements that can be processed mathematically, represent the situational problem by using a mathematical model, work out a mathematical solution, validate this solution and share the information related to the situational problem and the proposed solution. To do this, they must rely on their knowledge, imagination and sense of curiosity.

A situational problem involves one or more aspects of a complex of problems that must be solved using mathematical knowledge.

10. See Appendix B: Examples of Strategies for Exercising the Competencies.

Developing and using this competency provides an opportunity to establish intradisciplinary and interdisciplinary connections. For example, a solution that involves statistical concepts such as correlation may be similar to another solution involving algebraic concepts such as dependency relationships. Parallels can be made between the solving of situational problems, the investigative process in science and technology, and the creative dynamic in arts education. Each of these approaches requires students to draw on their creativity and reasoning ability, to explore possible solutions and to derive and validate models. In addition, each approach involves some combination of theory and experience or intuition and strategic implementation.

Situational problems must give rise to a need for a solution or to a cognitive conflict.

The situational problems vary depending on the goals of the students and the teacher. In all cases, they must give rise to a cognitive conflict or a need for a solution, allow for the integration of different types of knowledge or involve making connections that promote the transfer of learning. They may call for both manual dexterity and intellectual abilities.

Through questions, arguments, reflection and discussion with their peers, students continue to learn cognitive and metacognitive strategies. They will use these strategies to plan a problem-solving approach, to predict the result of an action in light of a specific goal, to organize and prioritize information, to modify strategies and to apply them in new situations. Reviewing these cognitive and metacognitive strategies contributes to the development of this competency.

The heuristic approach involved in solving situational problems requires students to use different types of reasoning (e.g. inductive and analogical), particularly to explore possible solutions, implement strategies or develop a model. To do this, students may choose to use mathematical concepts and processes related to arithmetic, algebra, probability, statistics or geometry. They may have to revise this choice or use a different register of representation when working out the solution. Their ability to reason when making these choices helps them conceptualize mathematical objects, make connections, and expand the networks of concepts and processes needed to work out a solution. When students solve a situational problem, they must also validate their solution. They critically assess how they use the given information. Their ability to produce proofs can be a valuable asset in shaping this critical judgment.

When solving a situational problem, students share information related to its solution and compare their solution with those from other sources. They use their ability to communicate in mathematical language when sharing solutions. Such information sharing occurs throughout the problem-solving process, providing students with food for thought, enabling them to confirm possible solutions or redirect their efforts, and enriching their knowledge of mathematics. In addition, their communication skills will allow them to obtain relevant information from material or human resources and to adapt this information to the situation at hand. For example, students may search for documentation, carry out a laboratory experiment in a team, use the Internet, and make use of resources and services both inside and outside the school.

The development of this competency in Secondary Cycle Two is based on what students learned in Secondary Cycle One. They must use their ability to solve situational problems in new contexts and are presented with more elaborate situations. They learn new strategies and are more often required to use their ability to develop models.

The cross-curricular competencies *Adopts effective work methods, Uses information* and *Uses creativity* contribute to the development of the competency *Solves a situational problem.* They help students to visualize the task at hand and become familiar with it, to mobilize resources and to confidently explore different possibilities.

The following are examples of how the competency may be used in each branch of mathematics.

– In arithmetic and algebra, solving a situational problem requires that students use their number sense and operational sense and the relationships between these operations. Their understanding of a situational problem allows them to distinguish explicit and implicit information from unknown or missing information and to illustrate relationships using tables of values, algebraic expressions or graphical representations. When they explore possible solutions in order to derive a model, they employ different strategies such as looking for patterns or using systematic or guided tests. They use the concepts of equation, function and system to work out a solution in cases where abstraction, interpolation, extrapolation, optimization, decision-making or weighing options are necessary to successfully complete the task. Throughout this process, students manipulate, estimate, validate and interpret data and numerical expressions written in various notations, taking into account their relative value in each context.
– In statistics and probability, students use their understanding of data drawn from statistical reports or from random experiments to define and solve situational problems relating to these branches of mathematics. They use graphical representations and tables to illustrate a situational problem, to organize and analyze data, to facilitate enumeration and to calculate probabilities and statistical measures. They use the concepts of chance and random experiment to choose a representative population sample and to validate or invalidate certain commonly accepted predictions and conceptions. They exchange information with their peers about their solution by explaining their approach, choice of registers, decisions, recommendations or conclusions. In the reaction of their peers, they look for ideas that will help them evaluate the efficiency of their solution or the reliability of the study in question. At every step in the problem-solving process, they may use simulations when an experiment is difficult to perform. They also use the concept of correlation to determine the nature and strength of the relationship between two variables. Technological tools are useful in conducting these simulations and correlation studies.
– In geometry, when students decode a situational problem, they use their spatial and measurement sense to define the task at hand and explore possible solutions. They form a mental image of the figures that are part of the situational problem. They represent two- or three-dimensional objects in various ways by using geometry sets or software, as needed. In developing a solution that involves finding unknown lengths, areas or volumes, in order to optimize them if necessary, students use definitions, properties or relations in working with numerical and algebraic expressions. They structure and justify the steps in their approach by using properties and postulates. They make sure that the result is plausible, given the context, and express it in the appropriate unit of measure. In discussing their solutions, students expand their network of relations and strategies.

MATHEMATICS
KEY FEATURES OF COMPETENCY 1

SOLVES a situational problem

Decodes the elements that can be processed mathematically

- Derives relevant information from various registers of semiotic representation: linguistic, numerical, symbolic, iconic and graphical
- Identifies and describes the task to be performed by focusing on the question being asked or by formulating one or more questions
- Determines the networks of concepts and processes to be used
- Uses different decoding strategies

Represents the situational problem by using a mathematical model

- Makes use of similarities between different situational problems
- Associates the situational problem with one or more branches of mathematics
- Illustrates the model by using different registers of representation
- Recognizes the limits of using known models
- Explores possible solutions

Shares information related to the solution

- Takes into account the context, the elements of mathematical language and his/her audience
- Provides a comprehensible and structured explanation of his/her solution

Validates a solution

- Compares his/her result with the expected result
- Checks his/her solution
- Rectifies his/her solution, if necessary
- Evaluates his/her solution, solutions worked out by his/her peers and solutions from other sources and compares them
- Gives reasons why a strategy or solution is inappropriate

Works out a mathematical solution

- Describes and estimates, if necessary, the expected result by taking into account the type of information given in the problem
- Applies appropriate strategies based on networks of concepts and processes
- Works out a structured procedure
- Justifies the steps in his/her procedure

End-of-Cycle Outcomes

By the end of Secondary Cycle Two, students in all three options solve situational problems involving several steps. They know how to use various strategies to represent a situational problem, work out a solution and validate this solution. If necessary, they explore different possible solutions and use concepts and processes specific to one or more branches of mathematics. They provide a structured solution that includes a procedure and a final answer, and they are able to justify and outline the steps in their solution using mathematical language. Lastly, they know how to make proper use of the instruments (software or other tools) that are necessary or appropriate for solving a situational problem.

Evaluation Criteria

- Oral or written indication that the student has an appropriate understanding of the situational problem
- Mobilization of mathematical knowledge appropriate to the situational problem
- Development of a solution* appropriate to the situational problem
- Appropriate validation of the steps** in the solution

* The solution includes a procedure and a final answer.
** The mathematical model, operations, properties or relations involved.
Note: The work involved in validating the solution may not always be shown.

Development of the Competency *Solves a situational problem*

In developing this competency, students make progress in constructing the mathematical edifice and work with increasingly complex situational problems. Although this progress is reflected mainly in the branches of mathematics and the concepts used, the situational problems to be solved may be characterized by a number of other parameters.

In Secondary Cycle One, students solved situational problems involving a multistep problem-solving process. They interpreted data and used different registers of semiotic representation and strategies to arrive at a solution. They learned to validate and communicate their solution by using mathematical language. They developed their ability to think critically by comparing their solutions with those of their peers. They also became aware of their aptitudes and of certain attitudes that characterize their learning style.

Students must solve more complex situational problems that give them the opportunity to use what they have learned and to acquire new knowledge that generally involves one or more branches of mathematics.

In Secondary Cycle Two, students must solve more complex situational problems that give them the opportunity to use what they learned in Cycle One. These problems generally involve one or more branches of mathematics. They reflect the aims of mathematical activity, take into account the specific educational objective of the option concerned and incorporate the broad areas of learning.

When teachers plan their instructional approaches to ensure or assess the development of this competency within a given year or from one year to the next in the cycle, they take into account a certain number of parameters in order to develop, adapt or modify learning and evaluation situations or to adjust their complexity.

These parameters are associated with the students' awareness of the ways in which they approach their work, with the contexts and the conditions under which the work is carried out and with the resources required. Some of these parameters are common to all situations, regardless of the competency involved:

– students' familiarity with the context
– the scope of the concepts and processes involved
– translations between the registers of semiotic representation
– the existence of intradisciplinary or interdisciplinary links
– the degree of autonomy required of students in carrying out the task

More specifically, a situational problem may be characterized by the following parameters:

– the strategies (affective, cognitive, metacognitive[11] or resource management) to be used in drawing up a solution plan, implementing it and validating it
– the student's familiarity with the task at hand or the human and material resources required
– the number of constraints, the amount of data or number of variables involved
– the level of abstraction required in order to understand the situation
– the nature and form of the expected or potential result
– the number and nature of the steps involved in working out the solution
– the nature of the connections that must be made between the different branches of mathematics or between the concepts and processes within the same branch
– the specificity of the required models
– the registers of representation involved

These parameters do not necessarily change in a linear fashion. Alternation between simple and complex situations is therefore recommended in order to meet the learning aims for each year of the cycle.

The tables on the following three pages give a synopsis of the elements of the learning content associated with the situations in which students develop this competency. The information in each table pertains to one of the three options in the program and covers the three years of Cycle Two. Note that the mathematical concepts and processes prescribed for the development and application of the competency are shown for each year under the *Program Content* section. This section also describes the spirit that characterizes each option and provides specific details on the contexts to be used.

11. The school team may refer to Appendix B in planning and gradually implementing and developing metacognitive strategies throughout the cycle.

First Year of Cycle Two

In the first year of Cycle Two, students work with situational problems from which they must derive relevant information that can be presented verbally, algebraically, graphically or in a table of values. The situational problems may involve one or more branches of mathematics and in solving them, students may need to manipulate different representations of data and numerical or algebraic expressions. They may be required to take into account the relative value of numbers when interpreting the task at hand, approximating results and developing and validating their solutions. The situational problems may also require students to switch registers in order to decode the given information or to represent elements of the solution. They will involve the use of number and operation sense as well as proportional reasoning in devising strategies to work out solutions. They give students an opportunity to make sense of algebraic expressions and to show an understanding of dependency relationships when analyzing contexts and making decisions. They may require that students interpolate or extrapolate by modelling the situations according to the functions under study, or it may require them to represent, interpret and compare probability data by enumerating possibilities and calculating the probability of events involving discrete or continuous results. The resolution of situational problems may require students to organize data from a sample (regardless of whether or not they have collected the data themselves) in order to describe a population about which they will draw conclusions. It may involve analyzing distributions using the appropriate statistical measures, or it may entail critiquing existing studies. Situational problems pertaining to geometry require students to construct or represent geometric figures by using various procedures. They also involve using spatial and measurement sense, and their resolution will depend on a knowledge of different relations associated with geometric figures and will require the calculation of unknown measurements.

Cultural, Social and Technical Option

Second Year of Cycle Two

In the second year of Cycle Two, the situational problems are related to various aspects (e.g. economic, social, demographic, technical, scientific) of students' everyday life. These problems may be related to one or more branches of mathematics and require students to make decisions. They involve identifying and using relevant information that may be presented verbally, algebraically, graphically or in a table of values and considering different options in planning the purchase of goods and services, for instance.

Among other things, the situational problems require students to draw on their number and operation sense and to use proportional reasoning to validate solutions. They may also involve solving systems of linear equations in order to compare and analyze phenomena with a view to making choices. They provide students with opportunities to interpolate or extrapolate by modelling situations using real functions represented in different forms. Certain decisions involve probability data, and require students to enumerate possibilities or calculate the probability of events involving discrete or continuous results or to use mathematical expectation to determine the possibility of gains or losses. If students are required to describe a population and draw conclusions about it, the situational problems will involve organizing data and analyzing one- or two-variable distributions to determine statistical measures (correlation coefficient, measures of central tendency, of dispersion or of position). Other situations require students to use different relationships associated with geometric figures. Among other things, they involve finding unknown measurements (length, area, volume) based on different metric or trigonometric relations involving right triangles, or congruent, similar or decomposable figures.

Third Year of Cycle Two

In the third year of Cycle Two, the situational problems sometimes involve making decisions using various mathematical tools, using various procedures to determine the option that best represents the preferences of a given population, or planning, estimating, evaluating and calculating different elements associated with organizing space or designing an object (e.g. costs, quantities, space, revenue). These tasks may be performed with or without the use of technology.

Some situational problems involve making predictions by modelling situations using real functions, while others entail solving problems or modelling situations (e.g. scheduling, optimal path, critical path) by using the concepts and processes associated with graphs. Still others involve using linear programming in order to choose the best solution(s). Those that focus on planning or measurements such as length, area, volume and residual space provide students with opportunities to use optimization. Furthermore, certain situational problems require students to consider the dependency relationship between certain events in determining the conditional probability needed to make decisions. The situational problems related to geometry sometimes involve the use of concrete materials or appropriate software. They require students to draw on their knowledge of geometry in designing and constructing plans and objects. They also involve finding different measurements by using definitions, properties, formulas or postulates with respect to triangles or congruent, similar, decomposable or equivalent plane figures or solids.

First Year of Cycle Two

In the first year of Cycle Two, students work with situational problems from which they must derive relevant information that can be presented verbally, algebraically, graphically or in a table of values. The situational problems may involve one or more branches of mathematics and in solving them, students may need to manipulate different representations of data and numerical or algebraic expressions. They may be required to take into account the relative value of numbers when interpreting the task at hand, approximating results and developing and validating their solutions. The situational problems may also require students to switch registers in order to decode the given information or to represent elements of the solution. They will involve the use of number and operation sense as well as proportional reasoning in devising strategies to work out solutions. They give students an opportunity to make sense of algebraic expressions and to show an understanding of dependency relationships when analyzing contexts and making decisions. They may require that students interpolate or extrapolate by modelling the situations according to the functions under study, or it may require them to represent, interpret and compare probability data by enumerating possibilities and calculating the probability of events involving discrete or continuous results. The resolution of situational problems may require students to organize data from a sample (regardless of whether or not they have collected the data themselves) in order to describe a population about which they will draw conclusions. It may involve analyzing distributions using the appropriate statistical measures, or it may entail critiquing existing studies. Situational problems pertaining to geometry require students to construct or represent geometric figures by using various procedures. They also involve using spatial and measurement sense, and their resolution will depend on a knowledge of different relations associated with geometric figures and will require the calculation of unknown measurements.

Technical and Scientific Option

Second Year of Cycle Two

In the second year of Cycle Two, it should be possible to represent situational problems using models specific to each branch of mathematics. The situations deal with various aspects of economics and involve choosing the goods and services that meet predetermined objectives. More specifically, they introduce students to case studies and the search for optimal solutions. Some situational problems involve producing, analyzing or comparing the parts of a bid that require mathematical processing. They may require students to use their critical judgment in analyzing plans, algorithms or suggested solutions in order to assess their efficiency and, depending on the case, identify errors or anomalies, make corrections, suggest improvements or make recommendations. Lastly, when necessary, other situations involve using appropriate instruments to develop a solution, taking into account the instruments' level of precision in validating the solution.

Certain situations require students to distinguish between different families of functions and to use them to develop models. They may involve solving equations, inequalities or systems of equations in all branches of mathematics. In problems involving chance, students make decisions by calculating conditional probability or mathematical expectation. Making a situation fair or optimizing the amount of a gain or loss, as the case may be, involves modifying the parameters of the situation (e.g. the rules of a game, the amount of a gain, an event). Other situational problems involve one- or two-variable statistical distributions. The use of real functions makes it possible to interpolate or extrapolate when this is required by the graphical representations of the variables under study. Solving certain situational problems pertaining to geometry, which entails the representation or construction of plans (or objects) in accordance with certain specifications, requires students to use their spatial sense and understanding of measurements. They also involve creating models and looking for optimal solutions by using the concepts of straight line, distance and point of division.

Third Year of Cycle Two

In the third year of Cycle Two, the resolution of situational problems enhances and expands students' repertoire of strategies, which includes the case study method. These situational problems may involve making comparisons, suggesting corrections, working out advantageous or optimal solutions or making recommendations. They require the ability to formulate constructive criticism and to make informed decisions about issues relating to various fields, including such technical fields as graphic design, biology, physics and business administration. In addition, some situational problems require an understanding of how various instruments work or of how they are used. Together with the ability to process data, this understanding prepares students to use new instruments.

Other situational problems involve using relations or functions to create models, to interpolate, to extrapolate and, where required by the solution, to perform operations on functions. Some situational problems require students to use a combination of geometric and algebraic concepts and processes or to use vector representation in their solution. As well, some situational problems, for which students must suggest advantageous or optimal solutions, involve solving systems of equations and inequalities, constructing objects or figures, and using the concepts of equivalent figures, geometric loci, distance or relative position. To solve other problems, students must find measurements by using metric or trigonometric relations pertaining to triangles and circles. Lastly, other situational problems are solved by using statistical and probability concepts that students have learned in previous years.

First Year of Cycle Two

In the first year of Cycle Two, students work with situational problems from which they must derive relevant information that can be presented verbally, algebraically, graphically or in a table of values. The situational problems may involve one or more branches of mathematics and in solving them, students may need to manipulate different representations of data and numerical or algebraic expressions. They may be required to take into account the relative value of numbers when interpreting the task at hand, approximating results and developing and validating their solutions. The situational problems may also require students to switch registers in order to decode the given information or to represent elements of the solution. They will involve the use of number and operation sense as well as proportional reasoning in devising strategies to work out solutions. They give students an opportunity to make sense of algebraic expressions and to show an understanding of dependency relationships when analyzing contexts and making decisions. They may require that students interpolate or extrapolate by modelling the situations according to the functions under study, or it may require them to represent, interpret and compare probability data by enumerating possibilities and calculating the probability of events involving discrete or continuous results. The resolution of situational problems may require students to organize data from a sample (regardless of whether or not they have collected the data themselves) in order to describe a population about which they will draw conclusions. It may involve analyzing distributions using the appropriate statistical measures, or it may entail critiquing existing studies. Situational problems pertaining to geometry require students to construct or represent geometric figures by using various procedures. They also involve using spatial and measurement sense, and their resolution will depend on a knowledge of different relations associated with geometric figures and will require the calculation of unknown measurements.

Science Option

Second Year of Cycle Two

In the second year of Cycle Two, the situational problems cover the different branches of mathematics. Some may involve experiments, while others may be purely mathematical. They require students to identify data presented verbally, algebraically, graphically or in a table of values and to model situations with the aim of carrying out an in-depth analysis, discerning patterns, and interpolating or extrapolating. Many of the situational problems require students to use their ability to perform algebraic operations. They involve using a rigorous mathematical approach and deductive strategies in order to arrive at one or more solutions. In addition, the situational problems consist of tasks that require students to validate and, if necessary, correct their solution(s).

Some situational problems involving the organization and interpretation of statistical data must be represented by a linear correlation. They require students to create models by using first- or second-degree polynomial functions or greatest-integer functions. Many of them require students to use their number sense and knowledge of algebraic manipulations to solve various equations and inequalities as well as systems of two equations in two variables, with one equation being of the second degree. Some involve different relationships within geometric figures or finding unknown measurements by using congruent, similar or equivalent figures or trigonometry.

Third Year of Cycle Two

In the third year of Cycle Two, most of the situational problems involve natural phenomena. They give students an opportunity to create mathematical models stemming from scientific experiments, for instance. In the process, students are able to determine and characterize the constituent elements of the problem and to understand its causes, effects and repercussions. These situational problems promote the transfer of knowledge from one branch of mathematics to another and from mathematics to other subject areas.

Some situations require students to perform operations involving functions and to solve systems of equations or inequalities. Furthermore, when students must rectify the steps in a solution, they must give reasons for the changes they make. Lastly, situational problems that involve geometry call into play the concepts and processes associated with conics and vectors in order to represent and analyze different phenomena.

COMPETENCY 2 Uses mathematical reasoning

Mathematical reasoning may be regarded rather schematically as the exercise of a combination of two faculties, which we may call intuition and ingenuity.
Alan Turing

Focus of the Competency

Using mathematical reasoning is an intellectual activity that entails a particular way of approaching a situation. It involves making conjectures and assessing, justifying or refuting a proposition by applying an organized body of mathematical knowledge. When students use mathematical reasoning, they examine a situation, determine how they will deal with it and organize their thinking by using inductive and deductive approaches, among others. This competency is essential to a range of mathematical activities.

In developing and exercising this competency, students must make conjectures, construct and use networks of mathematical concepts and processes, and validate these conjectures by drawing up proofs. Their thinking draws on different types of reasoning that shape their critical judgment and develop their ability to conceptualize as well as their willingness to understand and to justify.

A situation involving applications entails one or more implicit or explicit conjectures (relationships, statements, opinions, conclusions, etc.) that must be discovered, explained, generalized, proved or refuted using mathematical knowledge.

Mathematical reasoning involves a heuristic approach that is often implicit; it is a mental process. This approach plays a fundamental role in students' intellectual development, notably with respect to their analytical ability. Reasoning requires students to look for patterns and to describe, combine, invent or visualize. They will be all the more willing to engage in situations involving applications if they feel a need to discover, verify, explain, justify, convince, systematize or generalize. They will use their capacity to reason all the more readily if they wish to break a deadlock or feel the need to prove a conjecture. They must also recognize the value of examining information to define the relevant elements in the situation. To this end, students draw on their networks of concepts and processes and may use a variety of materials or consult different reference works, their peers or resource persons. This competency also involves using types of reasoning specific to each branch of mathematics as well as more general types of reasoning such as:

– inductive reasoning, which involves generalizing on the basis of individual cases
– analogical reasoning, which involves making comparisons based on similarities in order to draw conclusions
– deductive reasoning, which involves a series of propositions that lead to conclusions based on principles that are considered to be true; this type of reasoning also includes proof by exhaustion and proof by contradiction
– refutation using counterexamples, which involves disproving a conjecture without stating what is true

These types of reasoning lead to conclusions that are probable, plausible or definite, depending on the argument and branches of mathematics used to validate the conjecture in question. Students should not be asked to carry out tasks that call for only one specific type of reasoning, but should be required to work with situations in which they can use all of these types of reasoning.

The process involved in arriving at a conclusion is generally nonlinear; it involves doubt, dead ends, contradictions, working backwards, and so on. Whether consciously or unconsciously, students switch from one type of reasoning to another and adjust their approach, if necessary. The oral or written presentation of a solution is the most concrete sign of students' reasoning providing explicit evidence of their approach.

Constructing proofs remains one of the key features of the competency *Uses mathematical reasoning* that applies to all branches of mathematics. A proof is based on what is assumed to be true by a given community (e.g. the class, the general population, the mathematics community, the scientific community) and takes into account the audience and the degree of rigour required in an argument. A proof makes it possible to validate conjectures.

Students must learn to distinguish between reasoning and a mathematical proof, which is the codified presentation of that reasoning. Writing out the proof is therefore the last step in the process of validating a conjecture. This written presentation can be described as an explanation or a formal proof, depending on the student's approach.

The other two subject-specific competencies are crucial in exercising this competency. Students draw on certain key features of the competency *Solves a situational problem* when deciding on the truth value of a conjecture, constructing a proof, using appropriate elements of their knowledge to persuade others of the efficiency of a solution, organizing or structuring their efforts and approach, and justifying their viewpoints or decisions. Furthermore, the strategies involved in decoding information, looking for similarities or differences, choosing a mathematical model, exploring possible solutions and validating solutions all contribute to the development of students' ability to reason.

The expression of the student's reasoning is a complex activity that cannot be carried out without using the competency *Communicates by using mathematical language.* Mathematical reasoning and oral or written language are inextricably linked. In presenting their mathematical reasoning, language is the tool students use to initiate a line of reasoning when decoding information and formulating a conjecture in an attempt to understand a situation. Language is also the object of reasoning, since it is the means by which students combine and manipulate the concepts associated with the conjecture. Lastly, language is the vehicle for reasoning, since it conveys the conclusion that results from a line of reasoning by meeting logical or dialogical criteria.[12] The thoroughness of the reasoning will vary as will the way it is presented, depending on whether the student must convince himself or herself, convince someone unfamiliar with the given situation or validate the solution empirically or theoretically.

The development of this competency in Secondary Cycle Two is based on what students learned in Secondary Cycle One. The learning situations and types of reasoning to be used are nonetheless more elaborate. The students construct more complex and extensive networks of concepts and processes. Their ability to explain, to justify and to make conjectures becomes more refined.

Furthermore, exercising the competency *Uses mathematical reasoning* implies that students are able to put information to use, form and express an opinion, harness various resources and manage the communication process. It therefore involves using a number of cross-curricular competencies (notably *Uses information, Exercises critical judgment* and *Communicates appropriately*), while contributing to their development.

The following are examples of how the competency may be used in each branch of mathematics.

– In all branches of mathematics, students use their number and operation sense to construct and apply their networks of concepts and processes. They use numbers written in a variety of notations, including interval notation. They employ proportional reasoning to construct or interpret plans and figures, to convert measurements, to construct statistical tables or to analyze probabilities.
– In using their algebraic reasoning, students explore and compare different possibilities and then justify their choices in this regard. They identify different relations and, depending on the objectives, use interpolation, extrapolation or optimization processes based on their understanding of dependency relationships and the concepts of function and inverse function. They use algebraic processes to derive laws, rules and properties, which in turn serve to validate conjectures. For example, this might involve using deductive reasoning to show that two expressions are equivalent.
– In probability theory, students continue to learn about the concept of chance, which they use in their reasoning by considering all the possibilities in cases involving discrete or continuous results. They ask questions about the relationship between events (independence, equiprobability, complementarity, incompatibility). They use different types of reasoning when determining the truth value of certain commonly accepted ideas. For example, they use inductive or deductive reasoning to confirm their conjectures empirically through experiments, simulations or the statistical analysis of collected data.

12. Criteria related to the discursive reasoning, such as the criteria for drafting an argument.

– In statistics, students reason on the basis of data that they identify as qualitative or quantitative. They use different types of reasoning when they develop or analyze a questionnaire, choose a sample and process collected data. These activities involve organizing data, choosing the best way to represent it, interpreting it using different measures (of central tendency, of dispersion and of position) and validating the resulting conclusions. They compare distributions in order to analyze certain conditions or to make choices. In the case of two-variable distributions, they interpret the correlation and are able to interpolate or extrapolate using a regression line. Lastly, they exercise critical judgment when they come to a qualitative or quantitative conclusion about whether or not there is a dependent or causal relationship between variables.
– In geometry, students use reasoning when they identify the characteristics and properties of figures and perform operations involving these figures. They use different types of mathematical reasoning when they construct figures and compare or calculate measurements, notably by using algebraic expressions. They deduce properties or unknown measurements in different contexts by using definitions and postulates. In some cases, they use an indirect proof to come to a conclusion about the existence of a property.

MATHEMATICS

KEY FEATURES OF COMPETENCY 2

Makes conjectures

– Analyzes the conditions of a given situation
– Organizes elements selected from the network of concepts and processes that pertains to the situation
– Becomes familiar with or formulates conjectures adapted to the situation
– If needed, evaluates the suitability of the stated conjectures and picks the best ones

Constructs and uses networks of mathematical concepts and processes

– Establishes organized and functional relationships between concepts and processes
– Derives laws, rules and properties
– Uses different networks of concepts and processes (e.g. relating to algebra, geometry, proportions)
– Uses different registers of semiotic representation
– Coordinates the elements of mathematical language and everyday language pertaining to these networks

USES mathematical reasoning

Constructs proofs

– Uses different types of reasoning (e.g. inductive, deductive, analogical, proof by exhaustion, proof by contradiction) to clarify, validate, adjust or refute conjectures
– Uses methods specific to the branches of mathematics involved
– Organizes the results of a procedure
– If necessary, improves a procedure by eliminating superfluous steps

End-of-Cycle Outcomes

By the end of Secondary Cycle Two, students in all three options use the different types of thinking specific to each branch of mathematics when addressing a given situation or phenomenon. They make conjectures, apply appropriate concepts and processes, and confirm or refute their conjectures by using various types of reasoning. They also validate conjectures by basing each step in their proof on concepts, processes, rules or postulates, which they express in an organized manner.

Evaluation Criteria

– Formulation of a conjecture appropriate to the situation
– Correct application of concepts and processes suited to the situation
– Proper implementation of mathematical reasoning suited to the situation
– Proper organization of the steps in a proof suited to the situation*
– Correct justification of the steps in a proof suited to the situation*

* According to the year or option.

Development of the Competency *Uses mathematical reasoning*

When they use mathematical reasoning, students derive laws and properties by observing patterns and then relate them to concepts and processes that will help them to justify their actions. Students work out the steps in their line of reasoning as the need to convince or prove something arises. They learn how to better explain and organize their reasoning and how to fine-tune their supporting arguments. Students are gradually expected to be ever more rigorous in constructing their proofs, but the required degree of rigour varies according to the option they choose.

In Secondary Cycle One, students developed the ability to ask questions and make conjectures relating to a variety of situations. They learned to validate these conjectures by using their networks of concepts and processes as well as the types of reasoning specific to the different branches of mathematics, by justifying the steps in their procedure, by referring to principles and definitions or by looking for counterexamples. They were introduced to some of the basic rules of deductive reasoning and learned to distinguish between properties that have been proved experimentally and those that have been deduced.

In Secondary Cycle Two, students continue to develop their ability to reason and to exercise critical judgment. They must work with different situations involving increasingly complex concepts and the need to specify their ideas and present arguments with a view to forming an opinion, making comparisons and choices, and deciding on a course of action in order to make decisions, which they will then implement and evaluate. Students continue to construct a mental and operational representation of networks of concepts and processes. They reapply the networks with which they are already familiar, deepen their understanding of these networks and develop new ones. They learn to generalize and to draw conclusions with respect to concepts and the relationships between them. To do this, they switch from one type of mathematical thinking to another (i.e. arithmetic, algebraic, probabilistic, statistical and geometric thinking) and combine them when necessary.

Students are presented with meaningful situations that relate to the broad areas of learning, among other things. These situations provide them with the opportunity to work with both implicit and explicit data, to distinguish between essential and secondary information and to identify the conditions that are either necessary or sufficient or both necessary and sufficient. They use certain logical connectors (e.g. *and, or, if … then, if and only if, not*). They formulate conjectures and validate them by using different types of proofs (pragmatic or intellectual, direct or indirect), depending on the context, and by implementing different types of reasoning. In this way, students learn to think by becoming aware of approaches that allow them to construct knowledge and by becoming familiar with the structure of the reasoning they use. They learn to abstract by referring to the concrete and to situations involving elements that can be generalized. They organize their knowledge and ideas with a constant concern for coherence and clarity. They gradually improve their ability to analyze and synthesize information.

Students learn to think by becoming aware of approaches that allow them to construct knowledge and by becoming familiar with the structure of the reasoning they use.

When teachers plan their instructional approaches to ensure or assess the development of this competency within a given year or from one year to the next in the cycle, they take into account a certain number of parameters in order to develop, adapt or modify learning and evaluation situations or to adjust their complexity.

These parameters are associated with the students' awareness of the ways in which they approach their work, with the contexts and the conditions under which the work is carried out and with the resources required. Some of these parameters are common to all situations, regardless of the competency involved:

- students' familiarity with the context
- the scope of the concepts and processes involved
- translations between the registers of semiotic representation
- the existence of intradisciplinary or interdisciplinary links
- the degree of autonomy required of students in carrying out the task

More specifically, a situation involving applications may be characterized by the following parameters:

– the strategies (affective, cognitive, metacognitive[13] or resource management) involved in the reasoning used
– the students' familiarity with the types of reasoning they must use
– the nature of the data (explicit, implicit or unknown) on the basis of which students must identify the essential, necessary or sufficient information and carry out their activities
– the scope of the conjecture formulated or to be formulated
– the type of proof required
– the number and nature of the steps involved in validating a conjecture, coming to a conclusion or making a decision
– the scope of the explanations or justifications required to meet the demands of the work involved
– the nature of the connections or relations involved between the various branches of mathematics or between the different networks of concepts specific to a given branch
– the level of abstraction involved in the mental and operational representation of the concepts used and in the translations between the different registers of semiotic representation

These parameters do not necessarily change in a linear fashion. Alternation between simple and complex situations is therefore recommended in order to meet the learning aims for each year of the cycle.

13. The school team may refer to Appendix B in planning and gradually implementing and developing metacognitive strategies throughout the cycle.

The tables on the following three pages give a synopsis of the elements of the learning content associated with the situations in which students develop this competency. The information in each table pertains to one of the three options in the program and covers the three years of Cycle Two. Note that the mathematical concepts and processes prescribed for the development and application of the competency are shown for each year under the *Program Content* section. This section also describes the spirit that characterizes each option and provides specific details on the contexts to be used.

First Year of Cycle Two

In the first year of Cycle Two, the situations involving applications require students to use different types of reasoning by drawing on concepts and processes specific to each branch of mathematics in order to illustrate, explain, justify or convince. They also involve using examples or counterexamples, when necessary, to analyze conjectures. They give students the opportunity to identify the main steps in their line of reasoning. Many of these situations involve using number and operation sense and proportional reasoning in order to develop more efficient types of reasoning in the other branches of mathematics. Some concrete situations require students to use algebraic expressions and dependency relationships in order to analyze the situation and to interpolate or extrapolate from the situation. Others involve comparing measures in order to describe and quantify probabilities and, as the case may be, using experimental or theoretical probabilities to predict and validate results. Statistical reasoning is used when the situations require students to conduct, compare or critically examine certain studies. This may involve analyzing data and justifying conclusions by applying suitable tools such as statistical measures and graphical representations. Some of these situations may also require students to draw on their networks of geometric concepts and processes to deduce unknown measurements or validate conjectures. Lastly, all the situations give students an opportunity to illustrate their reasoning using various forms of representation (e.g. words, symbols, graphs, tables of values, drawings), depending on the situation and the branch of mathematics involved. These situations also involve using an inductive approach or a short series of deductions as part of a proof.

Cultural, Social and Technical Option

Second Year of Cycle Two

In the second year of Cycle Two, when the situations involving applications require the development of a line of reasoning, students must use strategies, concepts and processes specific to each branch of mathematics and analyze conjectures, notably through the study of examples. Students validate their conjectures by using different types of reasoning that they outline in direct or indirect proofs, whereas they refute their conjectures by means of counterexamples. The situations require students to interpret, predict, interpolate, extrapolate, draw conclusions, make decisions, present arguments or determine measurements, by taking into account relevant constraints. These situations may lend themselves to analysis based on the comparison of graphical representations. As well, the addition of information may give rise to new interpretations.

The situations involving applications lead students to base their reasoning on data presented in the verbal, algebraic, graphical or tabular registers. These data, which come from different sources, can be interpreted by using algebraic expressions, estimating a line of regression or performing calculations related to probability, chance and mathematical expectation. The data may also be based on statistical measures and graphs of one- or two-variable distributions dealing with a given population, or may be derived from a study of geometric figures that involves related definitions, properties and representations. When interpreting and evaluating conjectures arising from realistic situations in order to make informed decisions, students use their number and operation sense, proportional reasoning and understanding of probability and statistical data. The decision may also be based on an analysis of different sources of bias, on an analysis of the effect of modifying certain elements or on the calculation of relevant measures. Lastly, some situations require students to validate certain conjectures through short deductions based on geometry concepts, for instance. This requires them to justify their choices and the steps in their procedure.

Third Year of Cycle Two

In the third year of Cycle Two, the suggested situations involving applications require the use of different types of mathematical thinking in order to justify, prove, convince, deduce, refute or draw conclusions. They may involve formulating conjectures about various items of data and about the best way to represent and optimize this data. For example, students may analyze these data in order to convey as accurately as possible the preferences or choices of a given population or sample, to maximize output or to minimize losses. These situations may lead students to make decisions by considering various economic, spatial or time constraints. In order to be properly represented and modelled or for conjectures to be made and validated, some situations involve illustrating a line of reasoning through the use of the most appropriate register of representation. They require students to organize justifications and solutions, to compare, evaluate and critically examine choices or procedures, and to construct proofs, as applicable.

The situations may involve formulating and analyzing conjectures concerning optimization, among other things, and are related to one or more branches of mathematics. They require students to justify their procedure for arriving at an optimal solution, or their reasons for rejecting a conjecture. They may require an explanation of the possible effects of modifying certain constraints. Generalizing situations, evaluating trends and extrapolating from these trends require algebraic thinking, among other things. Some situations involve using the concepts of mathematical expectation and conditional probability in order to make informed choices. Others require the use of various statistical tools in order to draw conclusions or critically assess a statistical study while taking into account different factors. Lastly, situations involving geometric reasoning may require students to determine how to represent a geometric figure or an object. They also involve using various geometric tools to deduce different measures by basing one's reasoning on definitions and postulates.

First Year of Cycle Two

In the first year of Cycle Two, the situations involving applications require students to use different types of reasoning by drawing on concepts and processes specific to each branch of mathematics in order to illustrate, explain, justify or convince. They also involve using examples or counterexamples, when necessary, to analyze conjectures. They give students the opportunity to identify the main steps in their line of reasoning. Many of these situations involve using number and operation sense and proportional reasoning in order to develop more efficient types of reasoning in the other branches of mathematics. Some concrete situations require students to use algebraic expressions and dependency relationships in order to analyze the situation and to interpolate or extrapolate from the situation. Others involve comparing measures in order to describe and quantify probabilities and, as the case may be, using experimental or theoretical probabilities to predict and validate results. Statistical reasoning is used when the situations require students to conduct, compare or critically examine certain studies. This may involve analyzing data and justifying conclusions by applying suitable tools such as statistical measures and graphical representations. Some of these situations may also require students to draw on their networks of geometric concepts and processes to deduce unknown measurements or validate conjectures. Lastly, all the situations give students an opportunity to illustrate their reasoning using various forms of representation (e.g. words, symbols, graphs, tables of values, drawings), depending on the situation and the branch of mathematics involved. These situations also involve using an inductive approach or a short series of deductions as part of a proof.

Technical and Scientific Option

Second Year of Cycle Two

In the second year of Cycle Two, various situations involving applications enable students to show their understanding of a concept or a process, and provide the possibility of using different types of reasoning in making connections among the different branches of mathematics. Students perform various mental operations related to comparisons, explorations, testing and simulations. These operations lead students to make conjectures, to interpret, to draw conclusions or to construct proofs. Some contexts require a mastery of concepts and processes in order for students to be able to apply a line of reasoning when comparing and commenting on solutions, identifying errors and anomalies, or proposing modifications as required. The suggested situations are conducive to the formulation of structured explanations or justifications that show the steps leading to particular conclusions. Other situations may require students to identify and analyze the structure of a line of reasoning used in another person's procedure.

Many situations require students to create models and to decide on a course of action. Whether the data are produced experimentally or are deduced from information associated with a given context, these situations provide students with an opportunity to determine the nature of the relationship among the data and to establish the characteristics of the different families of functions. They allow students to interpret and use certain parameters and to predict the effect that a change in their value will have on a situation. Some situations involve working with numerical and algebraic expressions as well as the concepts of equality or inequality to perform operations involving interpolation, extrapolation or optimization. Others involve conditional probability and require students to identify the dependency relationship between events. They may also draw on the concept of mathematical expectation in the validation of conjectures involving the concept of fairness or the optimization of gains or losses. In addition, some situations require students to justify their choices or their conclusions when carrying out a statistical study or involve discussing the extent to which a study is representative or reliable. Other situations require them to apply geometry concepts, in a Euclidean or Cartesian plane, in order to deduce measurements or determine optimal solutions.

Third Year of Cycle Two

In the third year of Cycle Two, the situations involving applications require students to use different mathematical models and strategies and to combine reasoning and creativity in order to overcome an obstacle. They provide students with the opportunity to use structured deductive reasoning and to become familiar with the rules relating to formal proofs. They make it necessary for students to present arguments supported by illustrations, explanations and justifications. They involve different types of proofs and different types of reasoning, including proof by exhaustion. The latter is used, among other things, in analyzing or carrying out case studies or in implementing a process of generalization to validate a conjecture. Students are given an opportunity to observe real-world situations and to make generalizations based on these situations. Lastly, in certain situations, students analyze data so they can derive the necessary and sufficient information they need to reach a conclusion, make a decision, determine the best approach, and optimize or adjust a solution.

In analyzing certain situations, students will be required to refer to a functional model. They may adapt this model to the context at hand by adding appropriate parameters as well as a domain and a range. For the purposes of analysis and decision making, these situations may involve transforming equations or inequalities, organizing them into systems and performing operations on functions using instruments or technology, if necessary. Other situations may require students to illustrate their line of reasoning by using Euclidean or analytic geometry in order to determine measurements, optimize distances, construct geometric loci, plot the relative positions of figures or justify recommendations. Still others involve using the concept of vectors, representations in the Cartesian plane and geometric concepts to validate conjectures.

First Year of Cycle Two

In the first year of Cycle Two, the situations involving applications require students to use different types of reasoning by drawing on concepts and processes specific to each branch of mathematics in order to illustrate, explain, justify or convince. They also involve using examples or counterexamples, when necessary, to analyze conjectures. They give students the opportunity to identify the main steps in their line of reasoning. Many of these situations involve using number and operation sense and proportional reasoning in order to develop more efficient types of reasoning in the other branches of mathematics. Some concrete situations require students to use algebraic expressions and dependency relationships in order to analyze the situation and to interpolate or extrapolate from the situation. Others involve comparing measures in order to describe and quantify probabilities and, as the case may be, using experimental or theoretical probabilities to predict and validate results. Statistical reasoning is used when the situations require students to conduct, compare or critically examine certain studies. This may involve analyzing data and justifying conclusions by applying suitable tools such as statistical measures and graphical representations. Some of these situations may also require students to draw on their networks of geometric concepts and processes to deduce unknown measurements or validate conjectures. Lastly, all the situations give students an opportunity to illustrate their reasoning using various forms of representation (e.g. words, symbols, graphs, tables of values, drawings), depending on the situation and the branch of mathematics involved. These situations also involve using an inductive approach or a short series of deductions as part of a proof.

Science Option

Second Year of Cycle Two

In the second year of Cycle Two, the situations involving applications enable students to establish structured and functional links between concepts and processes in different branches of mathematics. These situations may involve contexts that are real, realistic or purely mathematical and require that students use in-depth and well-articulated reasoning in making and validating conjectures. They require that students organize the steps in their procedure in a coherent, comprehensive and efficient manner so that the structure of their reasoning is apparent. They involve using analogical, inductive and deductive reasoning as well as applying knowledge related to algebra, geometry, probability and statistics.

Some situations lead students to construct formal proofs by harnessing different types of knowledge (e.g. properties and operations involving algebraic expressions). Some situations enable students to analyze a model by determining and interpreting the value of the related parameters. Situations that involve the concept of correlation require students to use a type of reasoning that, through an understanding of dependency relationships and an ability for abstraction, allows them to recognize cause-and-effect relationships. Other situations involve proportional and geometric reasoning so that students can use the different metric and trigonometric relations in triangles. Some situations require students to find unknown measurements in geometric figures, which may or may not arise from a similarity transformation, to support or refute a conjecture. In other situations, students make use of definitions, properties, relations and theorems to prove other conjectures. Lastly, some situations may require students to identify the structure of the reasoning used by others to analyze it, to critically assess it or describe it in other words.

Third Year of Cycle Two

In the third year of Cycle Two, the situations involving applications require students to use various types of reasoning and to apply networks of concepts and processes as well as the connections between these networks in the different branches of mathematics. These concepts and processes are used to predict, simulate, make and validate conjectures. These situations allow students to identify trends or patterns, to make generalizations, and to interpolate or extrapolate from them. They lead students to illustrate a line of reasoning, notably through the construction of proofs that involve a deductive approach.

Some situations lead students to construct models based on information expressed in terms of a system of equations or inequalities in order to validate or refute conjectures. Students may need to use one or more real functions to model some situations. Other situations involve working with radicals, exponents, and logarithms, applying the laws governing these real numbers or using different algebraic and trigonometric properties to prove the identity of an expression or to convert, expand or reduce expressions to equivalent forms. Lastly, other situations require a combination of different types of geometric and algebraic reasoning and involve the concepts of conics and vectors.

COMPETENCY 3 Communicates by using mathematical language

We have been too quick to say that mathematics is a simple language that, in its own way, expresses the results of observation. This language, more so than any other, is inseparable from thought. We cannot talk about mathematics without understanding mathematics.
Gaston Bachelard

Focus of the Competency

Communicating by using mathematical language involves learning how to use the constituent elements of this language and to coordinate them appropriately in order to interpret, produce and convey messages. In developing this competency, students will not only focus on the usual characteristics of a sound message (e.g. clarity and concision), but also learn to appreciate the importance of precision and rigour.

Using this competency gives students the opportunity to increase their understanding of mathematical concepts and processes and to consolidate their learning, since they must clarify their thoughts through the very means they use to express them. In developing this competency, students are required to perform the following set of synergistically related actions: interpreting, producing and conveying mathematical messages; and making adjustments in communicating a mathematical message.

Students can use different registers of semiotic representation, either implicitly or explicitly, to become familiar with mathematical objects, to construct a mental image of them and to give this image meaning.

Using this competency entails mastering and coordinating the elements of mathematical language, which for the most part refer to abstract objects. Mathematical concepts do not exist as concrete entities. For example, a circle is a mathematical object constituting an idealized version of a shape that is found in nature and that can be represented verbally, graphically or symbolically. Students can use different registers of semiotic representation, either implicitly or explicitly, to become familiar with these objects, to construct a mental image of them and to give this image meaning.

Mathematical language is complex, because it is composed of different types of languages. Students develop their ability to easily switch from one register to another.

Mathematical language is complex, because it is composed of different types of languages, including everyday language. Beginning in elementary school, students familiarize themselves with the elements of mathematical language, which consist of words, symbols, figures, graphical representations and notations. These registers of semiotic representation may be classified in different ways. In general, they are divided into the following categories: the verbal register, the register of figures, the graphic register, the tabular register (tables of values) and the symbolic register. Students must be able to choose the registers appropriate to the situation and to work with these registers, meaning that they must follow the related conventional and syntax rules, identify specific information associated with these registers and perform operations within each register. For example, students who solve an equation by transforming it remain within the symbolic register. In addition, through various activities, students must develop their ability to easily switch from one register to another, for instance from the verbal to the symbolic register and vice versa. Some conversions (e.g. switching from the graphic register to the symbolic register or from the symbolic register to the verbal register) cannot be carried out directly or spontaneously. Since these conversions involve complex mental activity, they deserve special attention.

In addition to developing language skills, students become aware of the different meanings[14] that words can have when they are used in everyday language and in various subjects, including mathematics. They also learn

14. See Appendix D: Coordinating Elements of Mathematical Language.

about the roles played by quantifiers (e.g. *all, some, a few, at most*) and logical connectors (*and, or, if … then, if and only if, not*). Since several definitions of terms and symbols become clearer as learning progresses, they should be emphasized to ensure that students understand what they mean, recognize their usefulness and feel the need to use them.

When students are presented with a situation orally or in writing, their understanding may be hampered by the complexity of the vocabulary or phraseology used. With the help of certain reading or listening strategies similar to those used in the language of instruction, students are able to determine the purpose of the message, clarify certain terms, define the task to be performed and identify the information needed for the work involved.

Students employ various forms of discourse: descriptive, explanatory, argumentative or demonstrative.

Students communicate by using mathematical language when they solve equations, use a variety of representations, write out the steps in an algorithm, validate a conjecture, formulate a problem or definition, or present a procedure orally or in writing. To this end, students employ various forms of discourse: descriptive, explanatory, argumentative or demonstrative. Students *describe*, for example, when stating their observations. They *explain* when giving examples, rewording a statement, clarifying relationships, making a list or illustrating their ideas. They *argue* when they adopt a position or attempt to persuade others of the merits of their approach or the validity of their choices. These arguments may represent truths, opinions or value judgments. By contrast, when students seek to answer questions such as "Is it true?" or "Why is this true?", they are endeavouring to *demonstrate* the truth of a conjecture. The arguments presented in these cases are based on recognized or validated principles, which students use in putting together their line of reasoning.

The situations involving communication call for rigorous communication, in which the meaning of mathematical codes, formulations, rules and conventions is clear. Oral communication should be emphasized when the goal is to enable students to immediately respond to and influence what is going on. However, written communication also offers a number of advantages. Various writing activities promote the development of cognitive functions that are involved, for example, in the imitation of a model, the transformation of thought, memorization, analysis and monitoring actions. When students draw up proofs, certain writing rules make it possible to better organize their procedure and clarify their reasoning. When summarizing their knowledge in writing, they determine for themselves the most important elements to be included in their summary. In validating what they have written by interacting with others, they learn to view their work from another perspective, which sometimes encourages them to make changes to it.

A situation involving communication entails the need to manage messages using mathematical knowledge in a context where the subject and purpose of the message as well as the target audience play an important role.

Communication is beneficial for all those who participate in discussions, if only because the exchange of information is mutually rewarding. It is especially useful to those conveying the message because the need to explain their understanding of a mathematical situation or concept to an audience helps them enhance and deepen that understanding.

This competency, which is closely linked to the conceptualization and explanation of knowledge, processes and procedures, is also developed together with the other two subject-specific competencies. When working with mathematical situations that involve communication, students must organize and qualify their ideas by using strategies and abilities learned through activities that involve explaining a line of reasoning or a procedure. These activities help students develop their language skills. As well, when they describe their conception of mathematical objects to their peers or to their teacher, they improve their ability to communicate. Furthermore, when they choose a type of discourse that suits the purpose of their message and their audience, students use the strategies they have developed to present or justify an argument, or to persuade their audience of the truth value of a conjecture. In addition, each time they interact with their peers and their teacher, they learn to communicate, to reflect on their own actions and to provide constructive criticism when assessing the work of their peers.

The development of this competency in Secondary Cycle Two is based on what students learned in Secondary Cycle One; the situations involving communication are nonetheless more elaborate than those in Secondary Cycle One. Students expand their mathematical vocabulary and become more adept at switching from one register of representation to another. In contexts calling for rigorous communication, they learn to interpret or convey information in accordance with the purpose of the message.

Furthermore, the cross-curricular competencies, more specifically *Uses information, Adopts effective work methods* and *Communicates appropriately,* help students develop the competency *Communicates by using mathematical language*. They enable students to become familiar with given information, to visualize the task at hand, to use various modes of communication and to make adjustments in communicating their message.

The following are examples of how the competency may be used in each branch of mathematics.

– In arithmetic and algebra, students communicate in mathematical language when interpreting or producing symbolic expressions (equations, inequalities, systems or functions) that are useful for modelling relationships between quantities. They represent the connections between the elements of a situation by using everyday language, symbolism, graphs or tables of values. They set out and justify their viewpoints and choices when making a decision or when explaining how a change in certain aspects of the data will affect the model under study. In the communication process, students use their sense of numbers, variables and operations and choose the mathematical elements, units and notations suitable for the message they are trying to convey.
– In statistics and probability, students are in a situation involving communication when they enumerate possibilities or calculate a probability using a representation. When they organize, represent, analyze and interpret data, they highlight certain information by choosing appropriate registers of representation. They represent the situation using diagrams or graphs, draw up a questionnaire and present their results. Depending on the context and the type of data involved, they provide explanations concerning their choice of samples, graphic registers and statistical measures (central tendency, position, dispersion and degree of dependency). They outline arguments or formulate justifications to account for the decisions made and the conclusions drawn.
– In geometry, students communicate when they construct a geometric figure or when they describe, interpret or explain the data or any hypotheses associated with a problem. For example, they describe the properties of a figure or represent a three-dimensional figure in two dimensions by reproducing it from different points of view. They represent figures using conventional notation or decode them to obtain information. They explain and justify the steps in their reasoning, especially when writing up a proof. They use definitions, properties and postulates to make their reasoning clear and coherent. When determining unknown measurements, they communicate by identifying congruence or similarity relationships, applying metric relations, using appropriate units of measure and producing or interpreting formulas.

MATHEMATICS

KEY FEATURES OF COMPETENCY 3

COMMUNICATES by using mathematical language

Interprets mathematical messages

- Identifies the subject of the message
- Distinguishes between the everyday and mathematical meanings of various terms
- Associates images, objects or concepts with mathematical terms and symbols
- Expresses information using a different register of semiotic representation
- Asks questions or consults other people or sources of information to improve his/her understanding of a message
- Shares or compares his/her understanding of the message with others by expressing his/her viewpoint
- Synthesizes information

Makes adjustments in communicating a mathematical message

- Ensures that the rules and conventions of mathematical language are observed
- Discusses the relevance of the selected concepts, processes and registers of semiotic representation, taking into account the subject and purpose of the message
- Assesses the appropriateness of the means used to ensure that the audience understands the message
- Adapts the message according to the reactions and questions of the audience as it is being conveyed, and clarifies it, when necesary

Produces and conveys mathematical messages

- Defines the subject and purpose of the message (e.g. to inform, to convince)
- Chooses the type of discourse (e.g. descriptive, explanatory, argumentative) and the registers of semiotic representation depending on the audience and the purpose of the message
- Consults different sources of information, when necessary
- Organizes his/her ideas and establishes a communication plan
- Expresses his/her ideas using everyday language and mathematical language appropriate to the message, taking into account rules and conventions

End-of-Cycle Outcomes

By the end of Secondary Cycle Two, students in all three options are able to produce and convey oral and written messages that are unambiguous, coherent and adapted to the situation and the audience. Students are also able to interpret and analyze a mathematical message, to assess it critically and to improve it so that it meets the requirements of the situation. They use their ability to decode, describe, translate, transpose, represent and schematize, taking into account the subject and purpose of the message. In all cases, they use mathematical language appropriately by drawing on different registers of semiotic representation to show their understanding of a concept or message.

Evaluation Criteria

- Correct translation of a mathematical concept or process into another register of semiotic representation
- Correct interpretation of a mathematical message involving at least two registers of semiotic representation
- Production of a message appropriate to the communication context
- Production of a message in keeping with the terminology, rules and conventions of mathematics

Development of the Competency *Communicates using mathematical language*

To communicate by using mathematical language, students must constantly draw on different registers of semiotic representation. As their learning progresses, they develop their ability to use different registers in a given activity, which enables them to choose those that will be the most useful in providing the required information.

In Secondary Cycle One, students interpreted and produced oral and written messages by using various registers of representation. They honed their ability to choose mathematical terms and symbols and made appropriate use of information from several sources. In discussions with their peers, they analyzed different points of view and adjusted their message, if necessary. They used justifications or arguments to make sure they were understood or to persuade others. They developed solutions by outlining their procedure.

The elements of mathematical language involved in developing and using this competency are related to the learning content for each branch of mathematics.

In Secondary Cycle Two, students work with more elaborate registers of semiotic representation than they did previously and switch between the different registers more frequently, as it is now both possible and desirable that they know how to do this. The elements of mathematical language involved in developing and using this competency are related to the learning content for each branch of mathematics. The development of this competency is given concrete expression through the suggested communication activities. For example, students are required to discuss questions of a mathematical nature, to critically assess a decision or an algorithm, to express their ideas, to depict their view of a situation and to present their problem-solving procedures and strategies. They must take into account the context in which they are conveying their message, which they must adapt to the medium they are using and to their audience. In this way, they develop their ability to observe and listen, to express and convey messages and to acquire and integrate knowledge.

Whether they are constructing their understanding of a mathematical message or producing one themselves (i.e. proofs, summaries, presentations of solutions, lab reports, research findings, journals, oral presentations, debates, reports, projects), students use the appropriate strategies as well as their knowledge of different types of discourse and registers of semiotic representation.

When teachers plan their instructional approaches to ensure or assess the development of this competency within a given year or from one year to the next in the cycle, they take into account a certain number of parameters in order to develop, adapt or modify learning and evaluation situations or to adjust their complexity.

These parameters are associated with the students' awareness of the ways in which they approach their work, with the contexts and the conditions under which the work is carried out and with the resources required. Some of these parameters are common to all situations, regardless of the competency involved:

- students' familiarity with the context
- the scope of the concepts and processes involved
- translations between the registers of semiotic representation
- the existence of intradisciplinary and interdisciplinary links
- the degree of autonomy required of students in carrying out the task

More specifically, a situation involving communication may be characterized by the following parameters:

- the strategies (affective, cognitive, metacognitive[15] or resource management) involved in conveying the message
- the type of work involved (e.g. lab report, proof, description of a situation)
- the expected level of quality of the written work
- students' familiarity with the audience
- the purpose of the message (to describe, inform, explain, convince)
- the type of discourse required to produce the message (depending on the context and purpose of the message)
- the number and nature of the steps involved in producing, organizing and structuring the message
- the presentation of the information to be explored, decoded and interpreted
- the terminology, symbolism and concepts used in the wording of the situation
- the scope of the rules of conformity, transformation or conversion in the registers of representation used in the situation

15. The school team may refer to Appendix B in planning and gradually implementing and developing metacognitive strategies throughout the cycle.

These parameters do not necessarily change in a linear fashion. Alternation between simple and complex situations is therefore recommended in order to meet the learning aims for each year of the cycle.

The tables on the following three pages give a synopsis of the elements of the learning content associated with the situations in which students develop this competency. The information in each table pertains to one of the three options in the program and covers the three years of Cycle Two. Note that the mathematical concepts and processes prescribed for the development and application of the competency are shown for each year under the *Program Content* section. This section also describes the spirit that characterizes each option and provides specific details on the contexts to be used.

First Year of Cycle Two

In the first year of Cycle Two, the situations involving communication require students to write a description, an explanation or a justification and to work in a given register of semiotic representation. They may involve transforming a numerical or algebraic expression into an equivalent expression or switching from one type of graph to another. They involve converting from one register to another to make generalizations, to obtain additional information or to support an explanation or a justification. They may require students to analyze graphical representations or tables of values in order to extract specific information and present related conclusions. Some situations involve numbers written in different notations, taking into account the units where appropriate. In some situations, students are given an oral or written description and asked to draw the corresponding graph or find one or more related algebraic expressions. Conversely, the situations may also require a description based on a graph or a table of values. Situations that call for enumeration and probability calculations may involve using an appropriate mode of representation (e.g. drawings, tables, tree diagrams). As regards statistics, they may require students to interpret different statistical measures along with information taken from drawings and geometric constructions. Other situations may involve the projection of three-dimensional figures into two dimensions. They may also require students to use their sense of space, measurement and proportions to describe and interpret situations related to geometric figures or the concept of similarity. Situations involving geometry may require students to use definitions, properties and postulates. Lastly, they may also involve explaining the graphs, procedures and solutions chosen.

Cultural, Social and Technical Option

Second Year of Cycle Two

In the second year of Cycle Two, the situations involving communication involve writing up a description, an explanation, an algorithm or a proof. They often entail analyzing graphical representations or tables of values related to given situations so that students can extract specific information and present related conclusions.

Students may be given an oral or written description and asked to convert it into one or more equivalent algebraic expressions (e.g. equations, inequalities, systems of linear equations or functions). The situations involve the use of numbers written in different notations, taking into account the units, where necessary. They may also require students to describe a context based on a graphical representation or a table of values. They may also involve using different registers to represent situations, notably when this entails enumeration or probability calculations by means of appropriate diagrams. In some cases, students must draft a questionnaire before gathering data or producing a statistical report. Other cases may involve identifying and interpreting different measures collected by the students or by others. They may also involve deriving information from drawings and geometric constructions. In situations related to geometry, students decode data appearing on a geometric figure or construct an object from a two-dimensional representation. The situations also require students to use their sense of space, measurement and proportions in describing and interpreting geometric figures and addressing situations that involve similarity or trigonometry.

Third Year of Cycle Two

In the third year of Cycle Two, the situations involving communication may entail presenting a plan or writing up algorithms, proofs or reports. They require students to use available resources to present or obtain information, make comparisons, draw conclusions and justify their choices. In such cases, students must make sure that the message is structured, that it takes into account various factors and that it is presented properly and supported by arguments. They give students an opportunity to use different forms of representation: diagrams, symbols, graphical representations, tables of values and vertex-edge graphs. They also give students a chance to use different registers of semiotic representation and to choose graphical representations and procedures to explain solutions.

Some situations allow students to translate different constraints into systems of inequalities and to draw graphs of these systems in order to illustrate the situation and explain the solution chosen. They also lead students to determine, discuss and explain the sources of bias that may affect the presentation of statistical data. They require students to use different types of representation such as Venn diagrams or tree diagrams in contexts that involve determining conditional probability. They also enable students to extract information from a written statement, a table of values, a graph, a figure, an object or a two-dimensional representation. They may also require students to draw a plan or construct a geometric figure or an object using various procedures.

First Year of Cycle Two

In the first year of Cycle Two, the situations involving communication require students to write a description, an explanation or a justification and to work in a given register of semiotic representation. They may involve transforming a numerical or algebraic expression into an equivalent expression or switching from one type of graph to another. They involve converting from one register to another to make generalizations, to obtain additional information or to support an explanation or a justification. They may require students to analyze graphical representations or tables of values in order to extract specific information and present related conclusions. Some situations involve numbers written in different notations, taking into account the units where appropriate. In some situations, students are given an oral or written description and asked to draw the corresponding graph or find one or more related algebraic expressions. Conversely, the situations may also require a description based on a graph or a table of values. Situations that call for enumeration and probability calculations may involve using an appropriate mode of representation (e.g. drawings, tables, tree diagrams). As regards statistics, they may require students to interpret different statistical measures along with information taken from drawings and geometric constructions. Other situations may involve the projection of three-dimensional figures into two dimensions. They may also require students to use their sense of space, measurement and proportions to describe and interpret situations related to geometric figures or the concept of similarity. Situations involving geometry may require students to use definitions, properties and postulates. Lastly, they may also involve explaining the graphs, procedures and solutions chosen.

Technical and Scientific Option

Second Year of Cycle Two

In the second year of Cycle Two, the situations involving communication give students the opportunity to convey information, a description, an explanation or an argument orally or in writing. They require students to write up an activity, a communication plan or a report on an experimental procedure (e.g. laboratory report, logbook). They involve the presentation of a bid or a short structured proof in which observations, connections and justifications are clearly expressed. When working with these situations, students must interpret information derived from representations of objects, geometric constructions, plans, algorithms or solutions. They may also be required to formulate their observations, express their viewpoint, share their opinions, suggest corrections or make recommendations. These situations involve choosing registers of representation that are appropriate to the branches of mathematics concerned and switching from one register to another in order to illustrate, describe, compare, inform or explain. They may also require students to reformulate in their own words procedures, lines of reasoning or algorithms from external sources.

The suggested situations involve the use of various types of representation (e.g. tree diagrams, graphical representations, tables of values or charts, words or algebraic expressions, diagrams or figures) in order to interpret, produce and convey messages. Venn diagrams associated with conditional probability and scatter plots in statistics enrich the registers of graphical representation used. Some situations require students to convey messages through the use of different symbols, types of notation, units, logical connectors, quantifiers or literal expressions in accordance with the rules and conventions of mathematical language.

Third Year of Cycle Two

In the third year of Cycle Two, in addition to the types of activities carried out the previous year, the situations involving communication may entail preparing case studies, producing summaries, developing proofs or making presentations. In these situations, the students themselves must define the subject and purpose of the messages conveyed or received. Students may choose the medium, type of discourse and registers of representation best suited to their audience and the purpose of their message. In some cases, students must switch from one register to another. Among other things, the situations are designed to help students develop a wide range of communication strategies that enable them to adjust the delivery of their message according to the reactions of the audience or to take into account new requirements. They provide students with opportunities to learn a language that combines everyday, mathematical, technical and scientific terms in an effective way.

The diversity and quality of the messages produced are increased by presenting students with situations that require them to use new types of representations that involve functional models, systems of inequalities, geometric loci and transformations as well as vectors and equivalent figures. They enable students to describe, symbolize, code, decode, explain or illustrate information related to geometric figures, plans, graphical representations, tables of values, objects, and so on. In these situations, students must combine different registers of representation, as needed, in order to produce a message in accordance with the notation, rules and conventions associated with mathematical language.

First Year of Cycle Two

In the first year of Cycle Two, the situations involving communication require students to write a description, an explanation or a justification and to work in a given register of semiotic representation. They may involve transforming a numerical or algebraic expression into an equivalent expression or switching from one type of graph to another. They involve converting from one register to another to make generalizations, to obtain additional information or to support an explanation or a justification. They may require students to analyze graphical representations or tables of values in order to extract specific information and present related conclusions. Some situations involve numbers written in different notations, taking into account the units where appropriate. In some situations, students are given an oral or written description and asked to draw the corresponding graph or find one or more related algebraic expressions. Conversely, the situations may also require a description based on a graph or a table of values. Situations that call for enumeration and probability calculations may involve using an appropriate mode of representation (e.g. drawings, tables, tree diagrams). As regards statistics, they may require students to interpret different statistical measures along with information taken from drawings and geometric constructions. Other situations may involve the projection of three-dimensional figures into two dimensions. They may also require students to use their sense of space, measurement and proportions to describe and interpret situations related to geometric figures or the concept of similarity. Situations involving geometry may require students to use definitions, properties and postulates. Lastly, they may also involve explaining the graphs, procedures and solutions chosen.

Science Option

Second Year of Cycle Two

In the second year of Cycle Two, the situations involving communication require students to use networks of concepts and processes in each branch of mathematics in order to obtain, present, represent or justify information, or to persuade and inform others. Whether the situations involve presenting the validation of conjectures, or explaining procedures or considerations concerning the results of experiments or results obtained in other contexts, they are designed to encourage students to express themselves in a structured manner by using mathematical language and taking into account the characteristics of the different registers of semiotic representation. In so doing, students are able to demonstrate their communication skills.

Some situations involve working with data in the same register of representation, notably when writing the rules of second-degree functions in standard, general or factored form, while others involve switching from one register to another. Situations that involve systems of equations and inequalities require students to describe and interpret information. In producing certain messages involving statistics, students will use the concept of linear correlation. Other situations are centred on metric or trigonometric relations so that students may describe the relationship between different measurements within the same figure.

Third Year of Cycle Two

In the third year of Cycle Two, the situations involving communication help students develop the ability to interpret mathematical messages related mainly to scientific contexts but also to purely mathematical contexts. They involve tasks that require students to explain procedures, justify a line of reasoning or write up proofs. They require students to produce structured messages and to choose an appropriate representation so that the audience can understand the message. A number of situations involve strategies for decoding information, converting data from one register of semiotic representation to another and interpreting a message. Some require students to adapt the message so that it reflects the intended purpose.

The elements of a mathematical message are consolidated through the use of different registers of semiotic representation. Thus, some situations require students to use real functions to represent data and make it possible to derive information that can be interpreted critically and in an organized manner. Among other things, they involve representations containing algebraic, trigonometric, exponential or logarithmic expressions. Other situations require students to interpret and construct tables and graphs to support an explanation. Lastly, a number of situations involve switching from one semiotic representation to another by using geometric concepts.

Program Content

Time was when all the parts of the subject were dissevered, when algebra, geometry, and arithmetic either lived apart or kept up cold relations of acquaintance confined to occasional calls upon one another; but that is now at an end; they are drawn together and are constantly becoming more and more intimately related and connected by a thousand fresh ties, and we may confidently look forward to a time when they shall form but one body with one soul.
James Joseph Sylvester

The range of learning situations should enable students to develop an informed, aesthetic and critical view of the world.

The Mathematics program content for Secondary Cycle Two was developed with a number of considerations in mind. In order to meet the educational needs of students, this content reflects the aims of mathematical activity; it promotes the development of mathematical thinking as well as subject-specific and cross-curricular competencies; and it allows the broad areas of learning to be incorporated into the learning activities in accordance with the focus of each option. Furthermore, the program content, together with the goals of developing mathematical literacy, learning to ask questions and identifying students' personal and career-related interests, should promote meaningful learning and enable students to develop an informed, aesthetic and critical view of the world.

The competencies and mathematical content in this program are closely related. The students' ability to transfer concepts and processes to various situations indicates the extent to which they have mastered these concepts and processes. This ability is therefore an important factor in the development of the competencies *Solves a situational problem, Uses mathematical reasoning* and *Communicates by using mathematical language.*

The students' ability to transfer concepts and processes to various situations indicates the extent to which they have mastered these concepts and processes. This ability is therefore an important factor in the development of competencies.

The compulsory mathematical content in Secondary Cycle Two involves a set of resources that are essential for developing and using the competencies associated with this subject. The *concepts* and *processes* related to each branch of mathematics are presented first. Concepts are the mathematical objects under study and processes are the actions that make it possible to construct, develop and use these concepts. Only the new concepts and processes to be introduced each year appear in the following tables. It goes without saying that the learning content is not limited to these new concepts and processes, since students are required to use what they have learned in previous years.

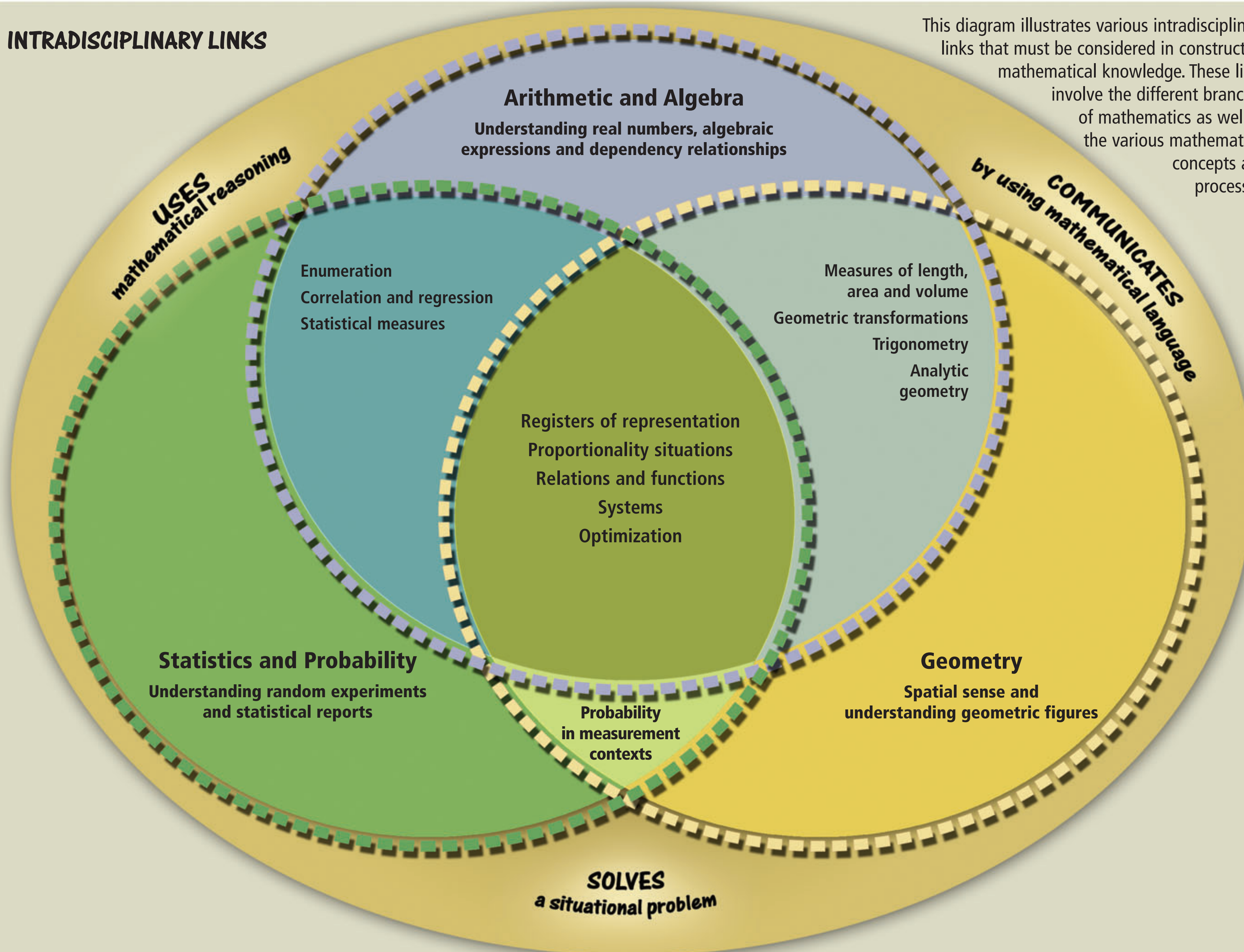
These tables are followed by descriptions of *learning processes* that define the scope and purpose of these concepts and processes. The program content can be viewed in either a linear fashion, owing to the sequence of prerequisites, or as a network of connections between the different branches of mathematics and between these branches and other subjects. The elements of the learning content should be regarded as symbiotic because the branches of mathematics interconnect and build on one another in such a way that the principles associated with one branch can be useful in understanding the principles pertaining to another branch. For its part, mathematical language consists of terms, notation, symbols and different registers of semiotic representation that students must master in order to exercise the competency *Communicates by using mathematical language.*

The sections on *Cultural References* provide various suggestions to help students to situate mathematical concepts in their social and historical context and to identify the problems and issues that led to their development. They focus on the concepts and processes associated with each option and illustrate the particular thrust of each option. They should enable students to better appreciate the influence of mathematics in their everyday lives, the needs it fulfills in society and the contributions that mathematicians have made to the development of mathematics and other subjects.

The Mathematics program content for Secondary Cycle Two is divided into five sections. The first provides an overview of the concepts studied in each branch of mathematics and a brief survey of their development over the three years of the cycle, taking into account the differences among the three options. The second section focuses on the first year of Cycle Two, covering the mathematical concepts and processes and the learning processes prescribed for that year as well as the related cultural references. The other three sections, one for each of the three options in the program, deal with the last two years of Cycle Two. As in the second section, they outline the mathematical concepts and processes and the learning processes prescribed for each of these two years, as well as the relevant cultural references.

INTRADISCIPLINARY LINKS

This diagram illustrates various intradisciplinary links that must be considered in constructing mathematical knowledge. These links involve the different branches of mathematics as well as the various mathematical concepts and processes.



The following tables show, for each branch of mathematics, the concepts introduced in each year of Cycle Two.

DEVELOPMENT OF THE MAIN CONCEPTS PERTAINING TO ARITHMETIC AND ALGEBRA IN CYCLE TWO

As part of their mathematical education, students develop different types of thinking, moving from arithmetic thinking to algebraic thinking. For example, they go from viewing the equals sign as a symbol that comes before a result to seeing it as an expression of an equivalence relation. Students thus have a better sense of numbers, operations and proportionality and develop their ability to model situations. The contexts they are presented with give rise to mental images that help them improve their understanding of numbers, operations and proportionality. Over the years, students hone their ability to evoke a situation by drawing on several registers of representation. For example, functions may be represented using graphs, tables or rules. Each of these representations conveys a specific point of view, and is complementary or equivalent to other types of representations.

SECONDARY CYCLE TWO			
First Year	– Real numbers: rational and irrational; cube and cube root – Inequality relation	**Relation, function and inverse** – Dependent variable and independent variable – Polynomial function of degree 0 or 1 and system of first-degree equations in two variables of the form $y = ax + b$, rational function of the form $f(x) = \frac{k}{x}$ or $xy = k$	
	***Cultural, Social and Technical* Option**	***Technical and Scientific* Option**	***Science* Option**
Second Year	**Algebraic expression** – First-degree inequality in two variables **Relation, function and inverse** – Real function: polynomial of degree less than 3, exponential, periodic, step, piecewise **System** – System of first-degree equations in two variables	**Arithmetic and algebraic expressions** – Real numbers: radicals, base 2 and base 10 powers – First-degree inequality in two variables **Relation, function and inverse** – Real function: second-degree polynomial (standard form), exponential, greatest integer, periodic, step, piecewise – Parameter **System** – System of first-degree equations in two variables	**Algebraic expression** – Algebraic identity, second-degree equation and inequality in one variable **Real function** – Step function (greatest integer); second-degree polynomial function (standard, general and factored forms) – Parameter **System** – System of first-degree equations in two variables – System composed of a first-degree equation and a second-degree equation in two variables
Third Year	**System** – System of first-degree inequalities in two variables	**Relation, function and inverse** – Real function: second-degree polynomial (general form), rational, sinusoidal (as well as the functions introduced the previous year) – Parameter – Operations on functions **System** – System of first-degree inequalities in two variables – System of equations and inequalities involving various functional models	**Arithmetic and algebraic expressions** – Real numbers: absolute value, radicals, exponents and logarithms **Relation, function and inverse** – Real function: absolute value, square root, rational, exponential, logarithmic, sinusoidal, tangent, piecewise – Operations on functions **System** – System of first-degree inequalities in two variables – System of second-degree equations (in relation to conics)

DEVELOPMENT OF THE MAIN CONCEPTS PERTAINING TO STATISTICS AND PROBABILITY IN CYCLE TWO

As part of their mathematical education, students develop their probabilistic and statistical thinking. With regard to understanding probabilities, students go from using subjective, often arbitrary, reasoning to reasoning based on various calculations. They become familiar with tools for processing the data collected, extracting information from that data and exercising critical judgment in order to identify potential sources of bias. Descriptive statistics provides students with a variety of concepts that help them to begin making inferences. By the end of secondary school, students are aware of the variability of samples as well as the limitations and constraints associated with population sampling.

SECONDARY CYCLE TWO			
First Year	– Discrete random variable and continuous random variable **One-variable distribution** – Sampling method: stratified, cluster – Graphs: histograms and box-and-whisker plots – Measures of central tendency: mode, median, weighted mean – Measure of dispersion: range of each part of a box-and-whisker plot		
	Cultural, Social and Technical Option	***Technical and Scientific Option***	***Science Option***
Second Year	– Subjective probability – Fairness: odds, mathematical expectation **One-variable distribution** – Measure of position: percentile – Measure of dispersion: mean deviation **Two-variable distribution** – Linear correlation: correlation coefficient and regression line	– Conditional probability – Fairness: odds, mathematical expectation **One-variable distribution** – Measures of dispersion: mean deviation, standard deviation **Two-variable distribution** – Linear and other correlation: correlation coefficient, regression line and curves related to the functional models studied	**Two-variable distribution** – Linear correlation: correlation coefficient and regression line
Third Year	– Conditional probability		

DEVELOPMENT OF THE MAIN CONCEPTS PERTAINING TO GEOMETRY AND GRAPHS IN CYCLE TWO

As part of their mathematical education, students go from using intuitive geometry, based on observation, to using deductive geometry. They discover the properties of figures by constructing them and explaining their constructions. Little by little, they stop relying on measurement and start to use deduction as the basis for their reasoning. By referring to data, initial hypotheses or accepted properties, students prove conjectures that are not evident, which are then used to prove new ones.

	SECONDARY CYCLE TWO		
First Year	**Solids** – Net, projection and perspective **Measurement** – Volume; SI units of volume; relationships between them		
	Cultural, Social and Technical Option	***Technical and Scientific Option***	***Science Option***
Second Year	**Analytic geometry** – Change: distance, slope, point of division – Straight line and half-plane: parallel and perpendicular lines **Measurement** – Relationships in triangles: sine, cosine, tangent, sine law and Hero's formula	**Analytic geometry** – Distance between two points – Coordinates of a point of division – Straight line: equation, slope, parallel and perpendicular lines, perpendicular bisectors **Measurement** – Metric and trigonometric relations in right triangles	**Equivalent figures** **Analytic geometry** – Straight lines and the distance between two points **Measurement** – Metric and trigonometric relations in triangles (sine, cosine, tangent, sine and cosine laws)
Third Year	**Equivalent figures**	**Equivalent figures** **Analytic geometry** – Geometric locus, relative position: plane loci involving lines or circles only, and conics – Standard unit circle – Vector (resultant and projection) **Measurement** – Metric relations in circles and trigonometric relations in triangles: sine and cosine laws	**Analytic geometry** – Standard unit circle and trigonometric identities – Vector – Conics: • parabola • circle, ellipse and hyperbola centred at the origin
	Graph – Degree, distance, path, circuit – Graph: directed, weighted		

Arithmetic and Algebra

Algebra reverses the relative importance of the factors in ordinary language. It is essentially a written language, and it endeavours to exemplify in its written structures the patterns which it is its purpose to convey. The pattern of the marks on paper is a particular instance of the pattern to be conveyed to thought. The algebraic method is our best approach to the expression of necessity, by reason of its reduction of accident to the ghostlike character of the real variable.

Alfred North Whitehead

In Secondary Cycle One, students develop their number and operation sense by working with numbers in decimal, fractional, exponential (integral exponent) and square root notation. They switch from one type of notation to another depending on the context. They identify relationships between operations as well as their properties. They follow the order of operations in performing sequences of operations involving no more than two levels of parentheses. They carry out operations, mentally or in writing, with numbers in decimal and fractional notation. They perform inverse operations and know properties of divisibility. They can locate numbers on a number line.

Students also begin to develop an understanding of proportionality, which is a central and unifying concept in Secondary Cycle One. They identify and represent proportionality situations in different ways: verbally, in tables of values, graphically, using rules. They learn the related concepts (i.e. ratio, proportion, rate, proportionality coefficient). They develop various multiplicative and additive strategies (e.g. unit-rate method, factor of change, proportionality coefficient, additive procedure) in working with situations that involve proportionality. By studying such situations, they start to understand dependency relationships.

In algebra, Secondary Cycle One students also develop their understanding of algebraic expressions by adding and subtracting them. They multiply and divide these expressions by a constant, multiply first-degree monomials and divide monomials by a constant. They write out and solve first-degree equations with one unknown that are expressed in the form $ax + b = cx + d$ and validate the resulting solution by substitution. They construct algebraic expressions from various situations. They evaluate algebraic expressions numerically and produce equivalent expressions. They use a graph to provide a comprehensive representation of a situation.

In Secondary Cycle Two, students reapply some of the concepts and processes they acquired in Secondary Cycle One and learn more about them. These concepts and processes serve as a springboard to new learning and enable students to make connections between various situations that are more complex than those in Secondary Cycle One. The concepts associated with the different types of notation (fractional, decimal, exponential, percentage), the rule of signs, operations, proportional reasoning, algebraic expressions and the meaning of equality are thus reapplied. The processes involved in using different ways of writing numbers, switching from one type of representation to another, evaluating an expression numerically and observing patterns are likewise brought into play and consolidated during Secondary Cycle Two.

To complete their basic education, students construct and master the following concepts and processes:

Understanding real numbers, algebraic expressions and dependency relationships	
Concepts	**Processes**
– Real numbers: rational and irrational • Cube and cube root – Inequality relation – Relation, function and inverse • Dependent variable and independent variable • Polynomial function of degree 0 or 1 - System of two first-degree equations in two variables (of the form $y = ax + b$) • Rational function of the form $f(x) = \frac{k}{x}$ or $xy = k$, $k \in \mathbb{Q}_+$	– Manipulating numerical and algebraic expressions • Using scientific notation in appropriate situations • Performing context-related calculations with integral exponents (rational base) and fractional exponents • Expanding and factoring algebraic expressions - Adding and subtracting algebraic expressions - Multiplying algebraic expressions of degree 0, 1 or 2 - Dividing algebraic expressions by a monomial ▪ Finding the common factor • Solving first-degree equations and inequalities in one variable - Validating and interpreting the solution – Analyzing situations • Observing, interpreting, describing and representing various concrete situations - Modelling a situation using a polynomial function of degree 0 or 1, or a rational function: verbally, algebraically, graphically and using a table of values ▪ Drawing a scatter plot to represent an experiment - Representing and interpreting the inverse function - Identifying the dependent variable and the independent variable based on the context - Observing patterns - Describing the properties of a function in context - Finding the rules, interpolation, extrapolation - Comparing situations ▪ Solving systems of first-degree equations in two variables by using tables of values, graphically or algebraically (by comparison), with or without the help of technology • Interpreting results

Note: In Secondary Cycle One, students did not systematically study sets of numbers. The program focused mainly on numbers written in decimal or fractional notation. In the first year of Secondary Cycle Two, students learn to distinguish between rational and irrational numbers and to represent various subsets of real numbers.

In manipulating numerical expressions, students learn to deduce the laws of exponents.

Students learn to make connections between exponential notation and radicals ($9^{\frac{1}{2}} = \sqrt{9}$, $8^{\frac{1}{3}} = \sqrt[3]{8}$).

Coefficients and constant terms in algebraic expressions are written in decimal or fractional notation. Numbers with a periodic decimal expansion or numbers that are easier to work with in fractional notation should not be converted into decimal notation. Similarly, radicals should be kept if there is no need to convert them.

Students learn to describe the properties of a function (domain, range, intervals within which the function is increasing or decreasing, extrema, sign and *x*- and *y*-intercepts). They identify them informally, always in relation with the context. The rule corresponding to a situation that can be represented by a polynomial function of degree 0 or 1 (linear function) or by a rational function may be found using an ordered pair and the rate of change or two ordered pairs. This rule is derived directly from the context, a table of values, a graph or another rule. In addition, students realize that, in some cases, ordered pairs of discrete values related to a situation are joined in order to derive a model.

Learning Processes

Arithmetic is taught in conjunction with the other branches of mathematics, which offer students various contexts for developing their number and operation sense. Intervals, either discrete or continuous sets of numbers, come into play when students group and interpret statistical or probability data, analyze situations by using the concepts of function and system as well as describe and interpret solution sets of inequalities. Geometry provides visual tools for understanding numbers (e.g. determining their order of magnitude by locating them on the number line, approximating their value by marking off ever more precise intervals on the number line or other methods). Scientific notation makes it easier to read and write both small and large numbers, and to understand prefixes such as *nano*, *micro*, *mega* and *giga*. In addition, it can be used to indicate the number of significant digits in a given number when necessary. Students also analyze the effect of operations performed on both members of an inequality. By developing an understanding of inequality relations, students expand and consolidate their understanding of equality relations and make connections between everyday language and mathematical language (e.g. *at most, no more than, at least, as much, as many*). Students have the opportunity to associate the meaning of logical connectors such as "and" and "or" with the operations of intersection and union, particularly when exploring sets of numbers, studying one of the variables of a given function or working in contexts involving constraints.

Algebra is a generalization tool that can be used to represent dependency relationships between quantities on the basis of observed patterns. The concept of inverse makes it possible to distinguish between the concepts of relation and function, among other things. Interpreting and representing a situation sometimes involves producing inverse models, depending on the independent variable selected. In the case of a first-degree polynomial function and a rational function, students are required to compare the rules, graphs and verbal descriptions of the dependency relationship associated with each function. Studying functions is an important part of the modelling process. In examining the graphic representation of an experiment, students come to realize that the resulting data do not always form a curve that corresponds exactly to a given mathematical model, because of handling or measurement errors or the level of precision of the instrument used, among other things. In experimenting with first-degree polynomial functions or rational functions, students find the curve that best fits the resulting scatter plot and perform interpolations or extrapolations. In addition, they analyze situations in which the rate of change differs depending on the interval involved, which enables them to make connections with broken-line graphs. Some situations involve the simultaneous examination of several functions that are translated into systems of linear equations. Analyzing the behaviour of a situation on either side of the point of intersection, if any, helps students choose a worthwhile solution, suggest modifications or formulate a new potential solution. They determine the solution of a system of equations graphically, by using a table of values, or algebraically by expressing the system in the form $ax + b = cx + d$ and then solving it.

The algebraic expressions that are added to the already known registers of semiotic representation allow students to look at situations from different but complementary perspectives. Beginning in the first year of Secondary Cycle Two, students learn to switch from one register to another without any restrictions. They refer to the context, properties and order of operations in attempting to understand algebraic expressions and manipulations. This new learning enables them to show that expressions are equivalent. Moreover, mathematizing situations by means of algebraic expressions, anticipating results or determining the value of an expression will help students develop their number and operation sense.

The use of technological tools makes it easier to explore and examine these relationships in greater depth and makes it possible to describe and explain them more fully. A number of exploration activities can be undertaken to promote the development of algebraic thinking. In this regard, Appendix E suggests avenues of exploration that will provide students with opportunities to make conjectures.

Statistics and Probability

Probability

Probabilities must be regarded as analogous to the measurement of physical magnitudes; that is to say, they can never be known exactly, but only within certain approximation.
Émile Borel

In Secondary Cycle One, students conduct random experiments involving one or more steps (with or without replacement, with or without order). They become familiar with the concept of event and explore the different types of events: certain, likely, unlikely, simple, complementary, compatible, incompatible, dependent and independent. They use different registers of representation (e.g. tree diagram, network, table) to enumerate possibilities. They calculate the probability of an event or compare theoretical and experimental probabilities. They develop their probabilistic thinking skills by formulating and validating conjectures. They analyze probability situations with a view to making predictions or decisions.

In Secondary Cycle Two, students reapply some of the concepts and processes they acquired in Secondary Cycle One and learn more about them. These concepts and processes serve as a springboard to new learning and enable students to make connections between various situations that are more complex than those in Secondary Cycle One. The concepts associated with random experiments involving one or more steps, and the concepts of event and theoretical and experimental probability are thus reapplied. The processes involved in enumerating possibilities and calculating probabilities are likewise brought into play and consolidated during Secondary Cycle Two.

To complete their basic education, students construct and master the following concepts and processes:

Understanding data from random experiments	
Concepts	**Processes**
– Discrete random variable and continuous random variable	– Interpreting probability data and making decisions related to the data • Enumerating possibilities and calculating probabilities in a variety of situations, including measurement contexts - Representing events using tables, tree diagrams, schematic drawings or geometric figures

Learning Processes

Probabilistic thinking skills are developed by learning to assign a probability (using an experimental or theoretical approach) to one or more events. In this regard, students should be encouraged to represent a situation in various ways. The registers of representation specific to this branch of mathematics are both comprehension and communication tools.[16] The situations explored may involve permutations, arrangements or combinations[17] and enable students to apply their reasoning by using tables or graphical representations, without necessarily employing counting formulas. When situations entail enumerating possibilities or calculating probabilities, they provide an opportunity to differentiate between the multiplicative and additive principles associated with logical connectors (*and, or*). Some situations involve events in a continuous sample space. When units of time, length, area or volume come into play, probability corresponds to a ratio of measurements, which requires students to use their knowledge of geometry in a probability context.

Probabilistic reasoning sometimes helps to reveal the erroneous nature of common perceptions or ideas about the probabilities associated with certain events. Sometimes ideas about representativeness, availability and equiprobability, and the confusion between probability and proportion raise interesting questions that give students the chance to formulate conjectures and validate them. In this regard, Appendix E suggests avenues of exploration that will provide students with opportunities to make conjectures.

16. See Appendix D.
17. Vocabulary (permutation, arrangement, combination) may be introduced in the first year of Cycle Two.

Statistics

... statistics is not a subset of mathematics, and calls for skills and judgment that are not exclusively mathematical. On the other hand, there is a large intersection between the two disciplines, statistical theory is serious mathematics, and most of the fundamental advances, even in applied statistics, have been made by mathematicians like R. A. Fisher.
Sir John Kingman

In Secondary Cycle One, students develop their understanding of statistics through studies based on sample surveys and censuses. They choose representative samples by using a simple or systematic random sampling method. They pay attention to the sources of bias that may affect the results. They characterize data, interpret them and represent them by selecting a suitable register of representation: table, circle graph, bar graph or broken-line graph. They use measures (mean and range) to derive information. They analyze results, compare distributions and formulate conclusions.

In Secondary Cycle Two, students reapply some of the concepts and processes they acquired in Secondary Cycle One and learn more about them. These concepts and processes serve as a springboard to new learning and enable students to make connections between various situations that are more complex than those in Secondary Cycle One. The concepts of population, sample and source of bias are thus reapplied. The processes involved in analyzing statistical data (e.g. constructing and interpreting tables and graphs, calculating the mean and the range) are likewise brought into play and consolidated during Secondary Cycle Two.

To complete their basic education, students construct and master the following concepts and processes:

<table>
<tr><th colspan="2">Understanding data from statistical reports</th></tr>
<tr><th>Concepts</th><th>Processes</th></tr>
<tr><td>– One-variable distribution
• Sampling methods: stratified, cluster
• Graphs: histogram and box-and-whisker plot
• Measures of central tendency: mode, median and weighted mean
• Measures of dispersion: range of each part of a box-and-whisker plot (including interquartile range)</td><td>– Analyzing situations involving a one-variable distribution, using appropriate tools, and making decisions related to these situations
• Organizing and selecting tools to collect, interpret and present the data and to account for the data:
- Constructing data tables: table of condensed data and table with data grouped into classes
- Drawing graphs: histogram and box-and-whisker plot
- Calculating measures of central tendency and of dispersion
• Comparing distributions
• Critically assessing the data collection method, the type of representation used or the results</td></tr>
<tr><td colspan="2">Note: In representations in which the data are divided into classes, the median is estimated using the middle value of the median class.</td></tr>
</table>

Learning Processes

Statistics allows students to make many useful connections with the other branches of mathematics as well as other subjects and the broad areas of learning. The statistical concepts that students learn may also help them to develop certain cross-curricular competencies, particularly the competency *Exercises critical judgment.* Using statistical concepts, students may, for example, draw conclusions or make informed decisions based on the results of a statistical report. The data collected, whether discrete or continuous, are represented by using various tools (tables, graphs, measures) that make it possible to synthesize information about a given population.

Statistics also allows students to compare different situations and to take a critical look at studies based on the use of samples. When analyzing a situation and drawing conclusions about it, students become aware of how the results and their interpretation may be influenced by such factors as the wording of the questions, the type and size of the sample, the interviewer's attitude, measurement errors, non-responses or the presentation of the study. They justify the measures of central tendency or of dispersion used in analyzing distributions or comparing populations. For example, the median is more suitable than the mean for representing a distribution that includes one or more outliers. In addition, certain situations may require students to make conjectures about how the value of measures of central tendency is affected by adding a constant to each value in the distribution, subtracting a constant from each value in the distribution, multiplying or dividing each value by a constant, or adding or removing a value that may or may not be an outlier.

Students work with a greater variety of more complex graphs. The graphs introduced previously represented data or relationships between the frequency of a variable and the total number of values in the distribution. Unlike other graphs, the box-and-whisker plot represents a "five-point summary" rather than displaying the data values individually. It allows students to visualize the distribution and to recognize and further examine the concepts of range and dispersion, thus laying the foundation for the study of mean deviation, variance and standard deviation. In addition, because the box-and-whisker plot depicts both a measure of central tendency (the median) and measures of dispersion (interquartile range, range), it makes it easier to compare distributions and explain these comparisons.

Geometry

If they would, for example, praise the Beauty of a Woman, or any other Animal, they describe it by Rhombs, Circles, Parallelograms, Ellipses, and other geometrical terms …
Jonathan Swift

In Secondary Cycle One, students continue to develop their spatial sense and expand their network of concepts and processes related to geometric figures. They learn about different concepts related to plane figures (polygon and circle): main segments and lines (bisector, perpendicular bisector, median, altitude, radius, diameter, chord), arcs and central angles. They identify angles based on how they are related to one another: vertically opposite, adjacent, alternate interior, alternate exterior, corresponding, complementary and supplementary.

Students in Secondary Cycle One recognize various convex polyhedrons and know how to represent their nets. They identify the different solids that make up a decomposable solid. They estimate and determine various measures of angles, lengths and areas and indicate them using the appropriate unit of measure. They construct the formulas for calculating these measures and predict the effect of changing a parameter in a formula. Through the constructions and transformations they perform, they identify properties and invariants. They refine their understanding of the concepts of congruence and similarity, which enables them to develop their spatial sense, to justify their procedures and to begin learning about deductive reasoning. They rely on definitions and properties in determining unknown measures.

In Secondary Cycle Two, students reapply some of the concepts and processes they acquired in Secondary Cycle One and learn more about them. These concepts and processes serve as a springboard to new learning and enable students to make connections between various situations that are more complex than those in Secondary Cycle One. The concepts of plane figures, solids, and congruent and similar figures are reapplied. The processes involved in finding the measures of angles, lengths (segment, perimeter and circumference) and areas are likewise brought into play and consolidated during Secondary Cycle Two.

To complete their basic education, students construct and master the following concepts and processes:

Spatial sense and geometric figures	
Concepts	**Processes**
– Solids • Net, projection and perspective • Measurement - Volume - Unit of measure for volumes - Relationships between SI units of volume, including measures of capacity	– Analyzing situations involving the properties of figures • Describing and constructing objects • Representing three-dimensional figures in the plane, using different procedures • Finding unknown measurements - Lengths ▪ Sides of a right triangle (Pythagorean theorem) ▪ Segments resulting from an isometry or a similarity transformation and segments in a plane figure or a solid - Areas ▪ Area of a sphere, lateral or total area of right cones and decomposable figures ▪ Area of figures resulting from a similarity transformation - Volumes ▪ Volume of solids that can be split into right prisms, right cylinders, right pyramids, right cones and spheres ▪ Volume of solids resulting from a similarity transformation - Choosing an appropriate unit of measure ▪ Converting between various units of measure (length, area, volume, capacity)

Learning Processes

By visualizing, manipulating and representing different objects, students continue to develop their spatial sense. In representing three-dimensional figures in two dimensions and, conversely, constructing solids from two-dimensional representations, students are able to explore various tools such as the net of a solid, orthogonal projections with different views, parallel projections (cavalier[18] and axonometric perspectives) or central projections (with one or two vanishing points). All of these representations are communication tools that can be used to interpret reality and that provide specific information. Students determine the best way to represent a three-dimensional model in two dimensions. They also learn that two-dimensional representations can be used to construct different solids. Exploration and hands-on activities help them understand the concept of measurement, construct formulas related to volume, consolidate the concept of area and apply the connections between geometric and algebraic concepts. In addition, students distinguish between the concepts of volume and capacity and use their proportional reasoning to perform various conversions.

Geometric thinking skills can be consolidated through the use of software and other instruments (e.g. blocks, lamps for studying shadows, dot paper or graph paper, scale models, assembly instructions) that are useful for constructing two- or three-dimensional figures, exploring their properties and developing manual skills. Students use their skills to interpret plans or specifications and to identify the perspectives used in such things as works of art or comic strips and in activities that involve decoding or creating optical illusions or so-called impossible figures. Appendix E suggests avenues of exploration that will help students develop their geometric reasoning skills by giving them the opportunity to make conjectures.

18. For constructions using cavalier perspective, students will normally be asked to draw the receding edges at angles of 30° or 45°. Other measures may be used, depending on the context.

Cultural References

Mathematical knowledge is universal and used everyday to interpret and understand reality and to make decisions. It enables individuals to participate in many spheres of human endeavour and to appreciate the contribution of this subject. The historical evolution of mathematics and the invention of certain instruments have been directly or indirectly related to the needs of different societies.

Mathematics has a rich history, and many mathematicians, scientists, artists and philosophers have contributed to its advancement. In activities related to the history of mathematics, students may notice that concepts and processes are often attributed to a particular mathematician despite the fact that they emerged through the efforts of a number of mathematicians, both men and women, from different eras (e.g. the Pythagorean theorem was already known in Babylonian times). In studying the contribution of women to the development of mathematics, students will learn that a number of women had difficulty achieving acceptance in the mathematics community.[19] By investigating the origin of certain words, students can make concepts and processes more meaningful and discover that researchers from many nations contributed to the development of mathematics. An epistemological dimension should therefore be incorporated into learning activities to provide a window on the past, the present and the future.

Arithmetic and Algebra

The composer opens the cage door for arithmetic,
the draftsman gives geometry its freedom.
Jean Cocteau

The development of mathematics has been shaped by the influence and contributions of different civilizations and cultures. For example, the Indians and Arabs shaped the development of mathematics in the Western world with regard to numeration systems, algebra and trigonometry. By examining these different contributions, students will be able to see and somewhat better understand how the set of real numbers was developed over time. This would involve examining questions such as the revolutionary significance of the introduction of zero, the reluctance to accept negative numbers and the crisis resulting from the incommensurability of $\sqrt{2}$.

Proportional reasoning has considerable currency in everyday life, and it is used in various occupations related to construction, the arts, health, tourism, administration and other fields. In addition, it has been studied by many different mathematicians throughout history (e.g. Thales, Eudoxus, the Pythagorians, Euler) to explain or represent phenomena that also relate to the arts (e.g. harmony in music, aesthetics in architecture).

The problem of infinity has provided food for thought through the ages. A discussion of the concept of infinity will give students the opportunity to visualize and reflect on the infinitely small or the infinitely large in sequences or intervals, for example.

In studying algebra, students may explore its origins by examining the general rules developed by Arab mathematicians in their efforts to solve problems. For instance, the work of Al-Khawarizmi in the 9th century, which dealt with the decimal number system and the solution of first- and second-degree equations, contributed to the development of the algebraic processes used today. The concept of function appeared around the 1700s. The idea of functionality gained currency in our society out of a concern for interpreting reality, particularly with regard to the study of motion and the calculation of time. Students may discover connections between the concept of function and the fields of music, ballistics, navigation, cartography or astronomy.

The different types of notation established by certain mathematicians make it possible to manipulate expressions more efficiently. In learning mathematical language, students will discover that the standardization of symbols took place over many centuries. Diophantos was one of the first to use symbols. It was not until the 15th century, however, that dedicated efforts were undertaken to create symbols and standardize them, though not without difficulty. François Viète made a major contribution in this regard. Today's students cannot help but notice the widespread use of various types of symbols (e.g. acronyms, logos, the short form of words, letters, numbers) and their impact on everyday life.

19. Some even had to pass themselves off as men so their work would be given consideration. Sophie Germain (alias Antoine-Auguste Le Blanc) is one such example.

Statistics and Probability

We define the art of conjecture, or stochastic art, as the art of evaluating as exactly as possible the probabilities of things, so that in our judgments and actions we can always base ourselves on what has been found to be the best, the most appropriate, the most certain, the best advised: this is the only object of the wisdom of the philosopher and the prudence of the statesman.
Jakob Bernoulli

Nowadays, it would be impossible to make progress in any area of science without the use of statistics and probability, among other things. In studying them, students will realize that these branches of mathematics are sciences that use precision and approximation to deal with uncertainty. Relatively new compared with geometry or algebra, statistics and probability were created and developed as a result of the need to understand various phenomena, to validate observations or intuitions and to predict an outcome in the more or less immediate future. They play a central role in our society as they facilitate decision making in many different areas.

Everyday life is filled with qualitative and quantitative data (e.g. graphs, rates, percentages, averages, predictions) in a variety of fields: health, employment, finance, sports, and so on. In addition, the learning situations involving these branches of mathematics can be readily linked to the students' immediate environment. For example, they will discover that the media often uses statistical and probability data. Whether they are looking at individual statistics in sports, economic comparisons or other information, they will be able to identify the most common registers of representation and classify them according to various criteria.

In the study of probability, the following question sometimes arises: "Does chance exist?" This question could give rise to a class debate that would ensure a common understanding of this concept. Students will discover that this concept has existed for a long time, even if the calculation of probabilities came into its own only in the 17th century with the contributions of Pascal, Fermat and the Bernoulli brothers. The Comte de Buffon laid the foundations for probabilities in geometric contexts through his analysis of the game of *Franc-Carreau* (open-tile or free-tile).

Through the ages, humans have collected data and conducted censuses for such purposes as taking inventory or evaluating and determining wealth or relative power. The need to develop statistical tools in order to extrapolate information from a population sample can, however, be traced back to the 17th century. Statistics was developed from observations about demographic data related to various fields such as public health (e.g. the work of Florence Nightingale in the 20th century). Students will learn that statistical processing has provided information on the increasing life expectancy of humans and that statistics is now used to make decisions related to politics, government, the economy, the environment and other fields. Any citizen with a basic ability to interpret statistical reports can obtain a host of information in a single glance.

Geometry

And since geometry is the right foundation of all painting, I have decided to teach its rudiments and principles to all youngsters eager for art …
Albrecht Dürer

Geometry is incorporated into a wide variety of situations in which students can identify objects that evoke geometric figures, related transformations and their properties. For example, they will observe geometric figures within structures and trajectories. They will analyze geometric transformations used in various areas of artistic expression such as tessellation (e.g. quilts, M.C. Escher engravings), weaving, Islamic art and musical works (e.g. canons, works of Johann Sebastian Bach).

Humans have always sought to represent the world in various ways. The study of perspective has provided some solutions, and perspective is used in various fields: geography,[20] media, computer graphics, design, engineering, architecture, photography, cinema, theatre, painting, and so on. During the Renaissance, the introduction of perspective revolutionized the arts. Students will discover that the artists of this period actually influenced mathematicians. Students may also be interested in the techniques and instruments devised

20. For example, scaled orthogonal projections provide contour lines, and the Mercator projection provides a representation of the world.

by Desargues, Dürer and Leonardo da Vinci. Various branches of geometry were developed during this period: projective geometry (Desargues), analytic geometry (Descartes) and, later on, descriptive geometry[21] (Monge). In addition, students will have the opportunity to learn about the explosion of knowledge that occurred in other areas of human endeavour during the Renaissance.

When developing their spatial and measurement sense, students may discover that Archimedes carried out a number of studies on area and volume (e.g. the crown problem solved by indirect proof). In addition, they will be able to visualize, develop or construct the thirteen Archimedean solids derived from the five Platonic solids: tetrahedron, hexahedron (cube), octahedron, dodecahedron, icosahedron. They will learn that five of the Archimedean solids were obtained by truncating Platonic solids, that Plato believed that the dodecahedron represented the universe and that the four other solids corresponded to the four elements fire, air, water and earth. Centuries later, Kepler proposed that the relative distances between the planets could be understood in terms of the Platonic solids enclosed within concentric spheres, but he subsequently concluded that the planets travel in elliptical orbits. Students may also have the chance to work with tangrams and polyominoes, including two- or three-dimensional pentominoes.

Students may also discover that the theorem attributed to Pythagoras was proved in many different ways (e.g. by Euclid, by the Chinese, by the Arabs, by U.S. president Garfield). A comparison of some of these proofs will reinforce the idea that more than one solution can be used to solve a problem and will encourage students to explore more than one possible avenue in validating a conjecture.

21. It is used mainly by designers, including architects and engineers.

Cultural, Social and Technical Option

Mathematics knows no races or geographic boundaries; for mathematics, the cultural world is one country.
David Hilbert

This section sets out some of the mathematical *concepts* and *processes* related to arithmetic, algebra, probability, statistics, geometry and graphs. These concepts and processes constitute the teaching and learning content specific to this option and are supplemented by the section on *learning processes* and *cultural references.* In addition, Appendix E outlines avenues of exploration that give students the opportunity to observe properties, make conjectures and validate or use them in exercising their competencies.

The *Cultural, Social and Technical* option is aimed at helping students to develop mathematical literacy[22] so that they can appreciate the connections between mathematics and the other aspects of culture as well as its contribution to the development of society. This option provides students with tools that help them to increase their capacity for analysis, to consider different possibilities, to make informed decisions, to support their reasoning, and to take a position with respect to various issues. It allows them to build on their basic education and to continue to develop their sense of citizenship. It helps them integrate into society and prepares them for higher education in different fields or for various types of vocational and technical training.

Besides continuing to develop their mathematical competencies and familiarizing themselves with new concepts and processes, students who choose this option will further their understanding of previously learned concepts. It is also important to allow them to use what they already know and to approach the learning content in a way that illustrates how mathematical ideas build on one another. Emphasis is placed on consolidating and integrating knowledge in a variety of activities: hands-on activities, exploration activities, simulations, games, research, presentations, debates, analysis of newspaper articles and advertising, meetings with resource persons, field trips in the city and the region, visits to museums, interpretation centres or companies, and so on. In situations that involve interpreting reality as well as making generalizations, predictions and decisions, students have an opportunity to use or develop their observational, design, managerial, optimization, decision-making, argumentative and other skills. They are generally required to carry out concrete and practical activities. Nevertheless, switching from the concrete to the abstract and using mathematical objects in concrete situations will help students to see their usefulness and encourage them to view various situations mathematically. Students will also use technology to represent or process large amounts of data and to make tedious calculations easier.

Students use their competencies and mathematical knowledge in different contexts related to the broad areas of learning. They are encouraged to look at the world from a critical, ethical and aesthetic point of view. They examine the social, economic, artistic, technical or, on occasion, scientific situations that they will encounter in their personal and working lives. For instance,

- health and well-being: living habits, nutrition, how the body works, health care, physical activity and sports
- consumer rights and responsibilities: personal finances, constraints related to production, costs related to consumption, design, advertising
- environmental awareness: planning (organization, plan, structure), resource management, biodiversity, pollution, population growth or decline

22. We are using the following definition of *mathematical literacy*: "Mathematical literacy is an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded mathematical judgements and to engage in mathematics, in ways that meet the needs of that individual's current and future life as a constructive, concerned and reflective citizen." This definition is used by the Organisation for Economic Co-operation and Development on p. 41 of the document entitled *Measuring Student Knowledge and Skills: A New Framework for Assessment*. This document was published in the framework of the Programme for International Student Assessment (PISA).

– career planning and entrepreneurship: design, planning, organization, market studies
– media literacy: presentation of information, comparison of presentations on the same topic, appreciation or creation of different works of art and media images
– citizenship and community life: social choices (voting procedures), equity and justice, cultural diversity, opinion polls, etc.

These situations are conducive to the use of proportional reasoning, number sense, dependency relationships, probability models, statistical tools, processes associated with graphs, spatial sense and measurement sense.

In addition, the cultural references suggested for this option involve historical and social aspects of the development of mathematics. They also provide examples of contexts in which students can develop their knowledge of concepts and processes.

Whether they are real, realistic, imaginary or purely mathematical, the situations examined and the mathematical knowledge that comes into play will help students develop a range of tools for observing phenomena, asking themselves questions, exercising their critical judgment, and using their intuition and creativity. They work with various situations in which they must apply their observational and analytical skills to obtain information, choose a mode of thought or a mathematical model, formulate conjectures, use strategies or consider different possibilities. They are also encouraged to base their line of reasoning or their approach on mathematical arguments or well-founded opinions.

Situational analysis and the processing of data are emphasized in the first year of Cycle Two. Students observe data and then represent, interpret and analyze them using mathematical tools in order to generalize, come to an assessment, draw conclusions or make decisions. In the second year, they consolidate what they have learned and use it to exercise their critical judgment, anticipate results and optimize situations, among other things. By the end of the cycle, they are able to explain, describe, develop arguments, deduce and justify. They begin learning how to put together a line of deductive reasoning although formal proofs are not emphasized. During the last year of the cycle, they are expected to use their mathematical knowledge, creativity and skills to complete a major independent assignment, which is described in the next section.

Students who choose this option are given many opportunities to learn about their world and are provided with an education that makes them aware of the many different skills and attitudes needed in our society. As a result, they develop competencies that will equip them to work effectively in a changing world and to act as informed citizens.

Independent assignment: comprehensive activity for integrating mathematical learning in the third year of Cycle Two

In the third year of Cycle Two, students who select the *Cultural, Social and Technical* option have the opportunity to integrate their mathematical knowledge by using their creativity, critical judgment and sense of entrepreneurship in carrying out a comprehensive activity. If necessary, they may start work on this activity at the beginning of the third year. This activity will give students an appreciation for the widespread use of mathematics in society, make them aware of the mathematical skills involved in performing various tasks and require them to demonstrate perseverance and autonomy. It therefore involves using all the competencies and all the branches of mathematics. The program stipulates that roughly 10 class hours are to be devoted to this activity.

During this last year, students enrolled in this option interpret real-world situations, generalize, predict and make decisions by using different optimization processes. In arithmetic and algebra, they look for the best or optimal solution. The calculation of probabilities and mathematical expectation helps them

Independent assignment: comprehensive activity for integrating mathematical learning in the third year of Cycle Two (cont.)

assess risks and make choices. Statistical tools allow them to process data and draw conclusions. By developing spatial sense and geometric knowledge, they are able to determine measurements and optimize certain design situations.

Examples

Given a real or fictitious budget, students can, among other things, design an object or a product, or contribute to the organization of a large-scale event (e.g. a graduation dance, a trip, a show, an exhibition). Where necessary, this may involve performing tasks such as:

- Planning and organizing the work involved
- Preparing plans, specifications and scale models
- Drawing up a budget
- Optimizing the shape and packaging
- Determining the target clientele
- Carrying out a market study on a number of prototypes
- Establishing projections, making predictions
- Defining production and marketing constraints
- Drawing up a business plan
- Comparing sources of funding or financing
- Creating a distribution network
- Submitting production or market study reports, and so on
- Conducting a promotional campaign
- Producing a balance sheet

Type of work that may be submitted

The final product can take different forms depending on the objectives involved. In all cases, however, it must include an explanation of the procedure for carrying out the activity.

- Examples of work that could be submitted: a mathematics logbook indicating how mathematical knowledge was used to carry out the comprehensive activity; a report; a summary; etc.

Note: The teacher may suggest different forums for presenting the results of the activity: in-class presentation, an exhibit or a one-on-one interview.

Expected outcomes with respect to the competencies

In carrying out their comprehensive activity, students use their ability to solve situational problems, their knowledge of the different branches of mathematics as well as their ability to represent, model and optimize situations. They demonstrate perseverance and autonomy in carrying out this activity. They use different strategies to plan, organize and develop the activity and to manage the various resources needed to carry it out.

Students use various strategies and different types of mathematical reasoning at every stage of the activity. They present their reasoning by highlighting the knowledge and connections involved, by formulating conjectures and, lastly, by justifying the steps they have followed and the conclusions they have drawn.

Students show their ability to communicate in mathematical language by using their knowledge of the different branches of mathematics. They use different means and strategies to present, adapt, manage and organize their message. They represent information using the most appropriate registers of semiotic representation. For instance, they may write a report in which they describe the different components of their comprehensive activity, the concepts and processes used, the steps involved and the conclusions they have drawn.

Evaluation

The work in this activity may be evaluated by the teacher, the student, his or her peers, or by all these people. Furthermore, the teacher may draw inspiration from the evaluation criteria outlined in the program to establish suitable criteria for evaluating the work in this activity. Students must nonetheless be made aware of these criteria. Assessment of the work in this activity will be taken into account in the evaluation of one or more competencies, as the case may be.

Arithmetic and Algebra

The interrelationships of algebra and geometry become more intelligible through the use of coordinates.
René Descartes

Building on what they learned in the first year of Cycle Two, students construct and master the following concepts and processes:

Understanding real numbers, algebraic expressions and dependency relationships	
Concepts introduced in the second year of Cycle Two – Algebraic expression • First-degree inequality in two variables – Relation, function and inverse • Real function: polynomial of degree less than 3, exponential, periodic, step, piecewise – System • System of first-degree equations in two variables	**Processes** – Analyzing situations related to economics (e.g. personal finances), social issues, technical or scientific contexts, or everyday life • Experimenting with real functions as well as observing, interpreting, describing and graphing them • Modelling a situation - Representing a situation using a table of values, algebraically in some cases or graphically, with or without the help of technology - Describing the properties of real functions using a graphical representation: domain, range, intervals within which functions are increasing or decreasing, extrema, sign, x-intercept and y-intercept • Comparing graphical representations • Interpolating and extrapolating data associated with a given situation, notably by means of a graph or technology (spreadsheet or graphing calculator) • Solving systems of first-degree equations in two variables • Making decisions if necessary, depending on the context
Concepts introduced in the third year of Cycle Two – System • System of first-degree inequalities in two variables - Polygon of constraints - Function to be optimized (*objective* function)	**Processes** – Analyzing and optimizing a situation and making decisions, using linear programming: • Representing a situation using a system of first-degree inequalities in two variables • Identifying and defining the function to be optimized • Drawing a bounded or unbounded polygon of constraints to represent the situation to be optimized • Calculating the coordinates of the vertices of the feasible region, using the systems of equations associated with the situation • Determining the best solution(s) for a particular situation, given a set of possibilities • Validating and interpreting the solution depending on the context • Changing the conditions associated with the situation to provide a more optimal solution, if necessary

Understanding real numbers, algebraic expressions and dependency relationships (cont.)

Note: Students use a graph, a table of values or technology in situations in which they must determine the value of the exponent.

Students differentiate, recognize and analyze various families of real functions and are presented with situations involving real functions expressed as follows: quadratic functions of the form $f(x) = ax^2$ and exponential functions of the form $f(x) = ab^x$ where $a \neq 0$ and $b > 0$. For the other functions, students may be given rules for which they are able to calculate values, sketch graphs and analyze properties, but without having to represent the situation algebraically.

In the case of situations involving personal finances, different aspects may be taken into account:
– types of income, such as types of compensation, salaries, commissions, contracts and gratuities
– different types of taxation, such as income tax, property tax and deductions at source
– types of financing, such as purchase options, personal loans, mortgages and financing costs
– the cost of certain utilities, such as the telephone or electricity

Learning Processes

In the *Cultural, Social and Technical* option, the arithmetic and algebraic concepts and processes learned previously serve as a springboard to new learning and can be used to make various connections. The concepts associated with real numbers, equivalent expressions, inequality relations, functions and systems as well as the processes related to proportional reasoning and numerical and algebraic manipulations are used and further developed. The situations allow students to consolidate and integrate these concepts and processes and to learn new ones that enable them to develop their algebraic thinking skills and their mathematical competencies. They must process data, derive the required information, determine models, analyze situations and make sound decisions based on mathematical arguments.

Second Year of Cycle Two

In the *Cultural, Social and Technical* option, the emphasis is on representing, analyzing and interpreting situations that can be expressed as functions. Students use different registers of semiotic representation in analyzing a given situation. In studying real functions, they learn to characterize the different types of dependency relationships between two quantities. Technology also allows them to explore, examine, describe and explain the relationships between two variables. They explore situations that do not necessarily involve linear relationships, such as exponential, rational, quadratic or step models. They observe patterns and distinguish between linear growth (arithmetic progression) and exponential growth (geometric progression) in situations involving population growth, for example. Faced with situations that require them to consider several functions or options simultaneously, students learn to represent them using systems of linear equations that can be solved in different ways: algebraically using the method of their choice (comparison, substitution or elimination), graphically or using a table of values. These situations may have no solution, one solution, several solutions or an infinite number of solutions. Students identify the characteristics of these different systems both in terms of their parameters and their graphical representation.

Situations involving personal finances (costs related to everyday purchases, earnings, income tax), exchange rates, and the depreciation or appreciation in the value of certain goods allow students to use their number sense and proportional reasoning, develop their algebraic thinking skills, and expand their knowledge of registers of representations and functional models (linear, exponential, step, piecewise). In given situations, students must be able to extrapolate, to make decisions reflecting the chosen model and different factors (e.g. influence of the period in question and the interest rate), and to justify their choices.

Third Year of Cycle Two

In the last year of Cycle Two, students draw on a whole range of concepts and processes associated with the different branches of mathematics. They use their ability to translate a situation into equations or inequalities and to work with algebraic expressions. They draw the Cartesian coordinate graph of the corresponding system by applying their knowledge of analytic geometry. In optimization situations, they determine the values of the decision variables in the function that optimizes (minimizes or maximizes) a situation involving a number of constraints. To facilitate decision making, they construct a model using linear programming. They represent the different constraints using a system of inequalities in two variables and algebraically define the function to be optimized. They represent the situation graphically, which allows them to observe the polygon of constraints or the feasible region. They use their knowledge of algebra in solving systems of equations in order to determine the coordinates of the related vertices.

During this final year, students are encouraged to apply their knowledge of arithmetic and algebra in different situations and, more particularly, in carrying out their comprehensive activity.

Probability

... in the small number of things that we can know with certainty, ... the principal means of getting to the truth ... are founded on probabilities.
Pierre Simon de Laplace

Building on what they learned in the first year of Cycle Two, students construct and master the following concepts and processes:

Understanding data from random experiments	
Concepts introduced in the second year of Cycle Two	**Processes**
– Subjective probability – Fairness • Odds • Mathematical expectation	– Analyzing probability data and making decisions related to the data • Distinguishing among theoretical, experimental and subjective probability • Distinguishing between probability and odds • Approximating and predicting results • Calculating and interpreting mathematical expectation
Concepts introduced in the third year of Cycle Two	**Processes**
– Conditional probability	– Analyzing probability data and making decisions related to the data • Distinguishing among mutually exclusive, non-mutually exclusive, independent and dependent events • Representing events, using tables, tree diagrams or Venn diagrams • Calculating conditional probability • Interpreting results – Making decisions concerning social choices • Counting and enumerating possibilities • Comparing and interpreting different voting procedures

Note: Factorial notation may be introduced in cases that involve counting. This type of notation makes it easier to write out certain operations, which can be performed efficiently using technology. Using a graphical representation, students should be able to enumerate situations involving arrangements and combinations. Finding and using counting formulas is not part of the curriculum for this option.

In cases that involve aggregating individual preferences (social choice theory), situations will be limited to no more than 4 "candidates." In particular, students compare and analyze majority rule, plurality voting, the Borda count, the Condorcet method, the elimination or runoff method and approval voting. See Appendix E.

Learning Processes

In the *Cultural, Social and Technical* option, the previously learned concepts and processes related to probability serve as a springboard to new learning and can be used to make various connections. The concepts of random variable (discrete and continuous), sample space, probability and event as well as the processes involved in enumerating possibilities and calculating probabilities are used and further developed in situations that ensure their consolidation and integration. In addition, the contexts, the new concepts and their related processes allow students to use their competencies and develop their probabilistic thinking skills in processing the data from random experiments, determining models, analyzing situations, developing their critical judgment with regard to various assertions, formulating predictions and making sound decisions based on mathematical arguments.

Second Year of Cycle Two

In the *Cultural, Social and Technical* option, students consolidate what they have learned and increase their knowledge of probability theory by carrying out experiments, among other things. They represent a situation by using a model that approximates the actual situation and analyze the collected data as though they were the actual data. Simulations can be carried out with or without the use of technology. Students distinguish between subjective probabilities[23] and theoretical or experimental probabilities in messages and speeches. They interpret various relationships and distinguish between them (i.e. the probability of an event and the *odds for* or the *odds against.*) They use the concept of mathematical expectation to determine whether a game is fair or the possibility of a gain or a loss. As a result of this analysis, they may change certain parameters to make the situation fair or to optimize a gain or a loss depending on their objectives.

Third Year of Cycle Two

When calculating the probability of compound events, students work with situations in which there is a restriction on the sample space. They construct the concept of conditional probability when calculating the probability of an event knowing that another event has already occurred. It is important that they properly define the situation and that they use a representation that will allow them to correctly interpret it so that they can determine the dependency relationship between the events. They also learn to distinguish between incompatible events and independent events. The different situations enable students to learn and use the language of sets. They use Venn diagrams, tree diagrams or schematic drawings to understand and communicate messages. They make connections with logical connectors, including "and" and "or." In order to develop their critical judgment, students learn to predict results, comment on behaviours or beliefs and make decisions that they explain or justify by using different probability concepts.

Mathematical models are also used in social, political and economic situations. Besides the models used to ensure the fair distribution[24] of different categories of members within representative organizations, other voting models or procedures involve aggregating individual preferences (social choice theory) in order to clarify the choices to be made in satisfying as many people as possible. By using their number and operation sense, their counting skills and proportional reasoning as well as percentages and weighted means, students will be able to compare the different models and analyze the difficulties they raise and the paradoxes that may arise.

In the last year of Cycle Two, students use their knowledge of probability theory in carrying out their comprehensive activity.

23. Subjective probability is used when it is impossible to calculate the theoretical or experimental probability. Such cases call for judgment, perceptiveness or experience. For example, weather reports involve the subjective evaluation of probabilities.
24. For example, this may involve the composition of a student council that is representative of students in every class, the distribution of seats in different parliaments, and the division or allocation of goods or resources in an inheritance.

Statistics

[Statistics are] the only tools by which an opening can be cut through the formidable thicket of difficulties that bars the path of those who pursue the Science of Man.
Sir Francis Galton

Building on what they learned in the first year of Cycle Two, students construct and master the following concepts and processes:

Understanding data from statistical reports	
Concepts introduced in the second year of Cycle Two	**Processes**
– One-variable distribution • Measure of position: percentile • Measure of dispersion: mean deviation – Two-variable distribution • Linear correlation - Correlation coefficient - Regression line	– Analyzing situations involving a one- or two-variable distribution, using appropriate tools, and making decisions related to these situations • Organizing and selecting certain tools to collect data related to a given population or sample and to present and account for the data - Constructing data tables, including tables for two-variable distributions - Drawing graphs: stem-and-leaf plots, scatter plots - Calculating measures of dispersion and of position - Approximating and interpreting the correlation coefficient - Representing the regression line using a rule or a graph - Interpolating or extrapolating using the regression line
Note: In the analysis and interpretation of distributions, the students' understanding of mean deviation should be considered more important than the calculations involved. In the study of correlation, analysis and communication should be considered more important than calculations. The correlation coefficient is estimated by using a graphical method (box method) and if the exact value is required, it is determined by using technology. Depending on the situation, students may conclude that certain correlation models are not linear.	

Learning Processes

In the *Cultural, Social and Technical* option, the statistical concepts and processes learned previously serve as a springboard to new learning and can be used to make various connections. The concepts of population and sample as well as the processes involved in collecting and processing data (sampling methods, graphs and statistical measures) are used and further developed in situations that ensure their consolidation and integration. In addition, the situations introduce new concepts and processes that allow students to continue developing their statistical thinking skills and mathematical competencies. Students use all they have learned in order to process data from statistical reports, ask relevant questions, determine models, present an analysis, develop their critical judgment, formulate predictions and make sound decisions based on mathematical arguments.

Second Year of Cycle Two

In the *Cultural, Social and Technical* option, students continue to learn about descriptive statistics and begin to make intuitive inferences. Like probability theory, statistics is a tool that can be helpful in making decisions. In order to answer questions of a practical or social nature, students collect data, organize them, represent them and determine different measures. They choose the graph that best represents the distribution and the information they want to present. They check whether the distribution includes outliers that could influence certain measures and their conclusions. As well, students look for biases that could affect the reliability of the study throughout the process. They are able to detect and correct any biases. To analyze and compare distributions, they observe their shape and use the appropriate measures of central tendency and of dispersion.[25] They identify the advantages and disadvantages of the different measures of dispersion: range, interquartile range and mean deviation.

Previously, the study of two-variable statistical distributions was introduced implicitly and experimentally through the use of scatter plots and the estimation of the regression line to help students develop an understanding of dependency relationships. The analysis of a scatter plot makes it possible to determine the correlation between the variables and to describe that correlation. Students produce a scatter plot on the basis of a two-variable distribution consisting of the results from a report they have been given or from an experiment they have conducted. They learn to interpret the correlation qualitatively: positive or negative, zero, strong or weak, perfect or imperfect. They determine the approximate value of the correlation coefficient by using a graphical method and, if necessary, calculate its exact value by using technology. It is important to analyze and interpret situations. Students become aware of the fact that a strong correlation does not necessarily mean that a causal link exists. This apparent relationship may indeed be coincidental or explained by a third factor.

In the case of linear correlation, students draw the line of best fit by taking outliers into account, determine the rule for this line and make extrapolations. They learn that the reliability of the interpolation or the extrapolation reflects the strength of the dependency relationship between the two variables. To draw this line or to find its equation, they can use the concepts of median or mean depending on whether the line of fit is determined using the median-median line method or the Mayer line method.[26] They may also use technology. However, the least-square method is not covered in this option.

Third Year of Cycle Two

In the last year of Cycle Two, students apply their knowledge of statistics in different situations and, more particularly, in carrying out their comprehensive activity.

25. Two statistical measures are generally used to describe a distribution: a measure of central tendency and a measure of dispersion. The two most frequently used measures in this regard are the mean and the standard deviation, but the median and the interquartile range are also used when the distribution is skewed. The standard deviation is not covered in the *Cultural, Social and Technical* option. Students use the mean deviation to analyze distributions.
26. The median-median line method involves dividing the data into three groups (the first and third groups must contain the same number of data values), finding the median point in each group and drawing the line that passes through the mean point of these three median points and that is parallel to the line that passes through the first and third median points. The Mayer line is constructed by dividing the data into two groups, calculating the coordinates of the mean points for each group and drawing the line that passes through these two points.

Geometry and Graphs

Ubi materia, ibi geometria
Where there is matter, there is geometry.
Johannes Kepler

Building on what they learned in the first year of Cycle Two, students construct and master the following concepts and processes:

Spatial sense and geometric figures	
Concepts introduced in the second year of Cycle Two – Analytic geometry • Change: distance, slope, point of division • Straight line and half-plane - Parallel and perpendicular lines – Measurement • Relations in triangles: sine, cosine, tangent, sine law, Hero's formula	**Processes** – Analyzing situations • Creating a graphical or algebraic model and representation of a situation, using a line or a half-plane • Finding unknown measurements or positions, using properties of figures or metric or trigonometric relations - Angles of a triangle - Lengths ▪ Side of a right triangle, altitude relative to the hypotenuse ▪ Side of a triangle ▪ Segment located in a Cartesian coordinate system or distance between two points - Areas ▪ Triangles and quadrilaterals - Coordinates of points
Concepts introduced in the third year of Cycle Two – Equivalent figures	**Processes** – Analyzing situations • Observing geometric transformations in the Cartesian coordinate system - Graphing and interpreting a rule • Finding unknown measurements (i.e. positions, angles, lengths, areas, volumes), using congruent, similar or equivalent figures, properties of figures, geometric transformations and metric or trigonometric relations • Optimizing results in different contexts such as the design of an object and situations involving economics - Comparing and calculating distances - Choosing the appropriate figure given a set of constraints

Note: The symmetric form of the equation of a line and the cosine law are not covered in the *Cultural, Social and Technical* option.
Translations, dilatations centred at the origin, reflections with respect to the *x*-axis and the *y*-axis as well as expansions and contractions are among the geometric transformations studied or represented. Rotations centred at the origin, where the angle of rotation is a multiple of 90°, are optional.

Learning Processes

In the *Cultural, Social and Technical* option, the previously learned concepts and processes related to measurement and geometry serve as a springboard to new learning and can be used to make various connections. The concepts associated with plane figures, solids, isometries, similarity transformations and projections and the processes involved in representing and constructing figures and determining unknown measurements are used and further developed in situations that ensure their consolidation and integration. In addition, the situations introduce new concepts and processes that allow students to continue developing their geometric thinking skills and mathematical competencies. Students use all they have learned for different purposes: to process data; to deal with various situations (related to consumption, the organization of space, the analysis of works of art or structures, design, construction, surveying, optimization or the use of instruments) that require them to use their practical, critical and aesthetic sense; to determine models and measurements; to analyze situations; and to make sound decisions based on mathematical arguments.

Second Year of Cycle Two

Through exploration based on what they learned previously, students are able to determine the minimum conditions required to conclude that triangles are congruent or similar. They can use these properties to find unknown measurements. By using proportional and geometric reasoning (e.g. the Pythagorean theorem and the properties of similar triangles), students are able to determine different measurements in triangles. By drawing the altitude from the right angle of a right triangle, they learn to recognize similar triangles and to establish proportions, thereby deducing metric relations in right triangles. Using similar triangles, students will become familiar with the trigonometric ratios in right triangles. Depending on the information provided, they can calculate the height of a triangle using the sine ratio and, by extension, determine its area. They can also do this using the sine law or Hero's formula.

Analytic geometry provides a link between geometry and algebra. It familiarizes students with the concept of change, which they will use to calculate the distance between two points, the slope and the coordinates of a point of division, among other things. They determine the equation of a straight line using the slope and one other point or two points. They explore the relative position of two straight lines in the Cartesian coordinate system and identify patterns. They use proportional reasoning and the concept of change to find the coordinates of a point that divides a segment in a given ratio: i.e. they add a fraction of the horizontal change to the *x*-coordinate and a fraction of the vertical change to the *y*-coordinate of the initial point.

The avenues of exploration given in Appendix E are examples of conjectures that students could be required to make after they have studied families of figures. They could examine these statements in order to exercise their reasoning in a geometric context. Although the properties studied do not necessarily have to be proved, they should represent conclusions that students will draw during exploration activities that require them to use their spatial sense and their knowledge of the properties of geometric transformations, among other things. These statements help students justify their procedure when solving a situational problem or using mathematical reasoning. Through situations related to this option, students hone their deductive reasoning ability by learning how to deduce certain properties using organized reasoning based on definitions or other established properties.

Third Year of Cycle Two

In the last year of Cycle Two, students use their knowledge of geometry to design and represent objects. They study and analyze a few geometric transformations in the Cartesian coordinate system. They discover the rules for finding the image of a point. They draw the image of a figure on the basis of a rule or predict the effect of the rule. They may be required to define a rule in order to create a transformation. In this way, they deepen their knowledge of the concepts of congruence, similarity and function, observe how images are programmed in computer graphic animation and perform different transformations using the figures they created.

Students use the concept of equivalent figures[27] to determine certain measurements. They derive the main properties of these figures and make connections between the total area and the volume of a solid. They compare equivalent figures and determine which one is most appropriate to meet certain objectives (e.g. to maximize or minimize space). For instance, in situations involving optimization, they determine the most economical shape for a container or package of a given volume, taking into account such factors as ease of storage. They may calculate the ratio between volume and total area and see a connection between the value of the ratio and the most economical shape. They analyze and interpret situations involving measuring instruments, packaging, photography, lamps and shadows, and so on. They are thus able to use their knowledge of geometry in different situations and, more particularly, in carrying out their comprehensive activity.

27. *Equivalent plane figures* are figures with the same area and *equivalent solids* are solids with the same volume.

Understanding data represented by means of graphs	
Concepts introduced in the third year of Cycle Two	**Processes**
– Graph • Degree, distance, path and circuit • Graph: directed, weighted	– Analyzing and optimizing situations involving the concept of a graph and making decisions related to these situations • Representing and modelling a situation using a graph that may or may not be directed, coloured or weighted (including trees) • Comparing different graphs • Finding Euler and Hamiltonian paths and circuits, a critical path, the shortest path, a tree of minimum or maximum values or the chromatic number
Note: Terms pertaining to graphs are introduced as they arise within situations. Students are not required to memorize a set of definitions. Properties are also introduced within exploration situations. Certain properties can be proved, as needed, by applying the properties of numbers. Some of the properties of graphs are given in Appendix E.	

Learning Processes

Third Year of Cycle Two

Graphs are simple, yet powerful concepts that play an important role in helping students develop and use their mathematical competencies. They represent the relationships between the elements of a structure. In using graphs, students learn how to model situations they encounter in their daily life or will face in the future. In learning about graph theory, students are introduced to a different type of reasoning. To draw a graph for a given situation, they must choose the elements that will be represented by vertices and those that will be represented by edges. The situations may involve different types of planning, distribution or communication networks, circuits, incompatibilities, localizations, strategies, and so on. Depending on the situation, students use different types of graphs: directed or undirected, coloured or not coloured, weighted or unweighted, including trees. To optimize certain situations, they find the critical path, use graph colouring, or determine trees of minimum value or the shortest path. For example, they may formulate conjectures concerning the minimum number of colours and use different types of reasoning to validate their conjectures, thereby distinguishing between a necessary condition and a sufficient condition. They develop their own algorithms and compare them with simple algorithms such as Dijkstra's algorithm for the shortest path. In addition, they can use graphs to represent or construct labyrinths or winning strategy games.[28] Using a graph representing the outcome of a winning strategy game, students can work backwards to determine which positions could lead to victory.

Students can use their knowledge of graphs in carrying out their comprehensive activity.

28. *Winning strategy games* involve two opponents who take turns playing; the game cannot end in a tie [e.g. the Game of Nim (matchstick game) and its variations, race to 20].

Cultural References

In most sciences one generation tears down what another has built, and what one has established, another undoes. In mathematics alone each generation adds a new storey to the old structure.
Hermann Hankel

Exposure to mathematical culture helps students develop competencies and master the concepts and processes specific to the different branches of mathematics. Mathematical knowledge, even the most basic, is a part of this culture. It serves to express the logical structure of things and phenomena and by this very fact is a tool for critical thinking. Familiarity with mathematical culture enables students to take part in various activities in any field and makes it possible to appreciate the contribution and widespread application of mathematics and to understand that mathematical knowledge results from the extensive work done by researchers with a passion for this subject, be they mathematicians, philosophers, physicists, artists or others. An epistemological dimension should therefore be incorporated into learning activities to provide a window on the past, the present and the future.

The cultural references associated with the *Cultural, Social and Technical* option provide students with the opportunity to understand and appreciate the use of mathematics in everyday life, its place in the evolution of humankind and the contribution of various researchers and thinkers to the development of this subject. Although the construction of the mathematical edifice has been marked by the influence and contribution of different civilizations and cultures,[29] students will realize that several mathematical concepts covered in this option are relatively new and that they do not all date back to antiquity. Mathematics is a science that is constantly evolving, regardless of the sphere of human activity in which it is used.

29. For example, the Chinese were already familiar with Pascal's triangle three centuries before Pascal formulated the concept.
30. Students will see that the vocabulary that is used comes from different fields. For example, the term "programming" comes from military language.

Arithmetic and Algebra

A mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas.
Godfrey Harold Hardy.

Proportional reasoning is widely used in everyday life and in different occupations in fields such as construction, the arts, health care, tourism and business administration. Observations of the dependency relationship between two quantities contributed to the development of the concept of function, which is used in such areas as navigation, astronomy and ballistics. Students will have the opportunity to discover its importance and to appreciate how different mathematicians, such as Oresme, Descartes and Fermat, contributed to the development of the concept of function and analytic geometry. Later, Thomas Malthus analyzed arithmetic and geometric progressions in his work on population growth and the growth of the food supply. With this idea in mind, students will be asked to make observations, comparisons or decisions regarding different instances of growth and decline in fields such as demography and finance.

Certain models have been useful for meeting different needs and making decisions. Operational research helps decision-makers make choices. Linear programming[30] is one of the most widely used models. The linear programming work of Leonid Kantorovich and T.J. Koopmans served as the basis for George Dantzig's development of the "simplex method" used to solve procurement problems during the Second World War. In addition to making connections with contemporary historical events, students will discover that operational research is used in areas such as economics, management, agriculture, computer science and environmental science.

With regard to different instruments, students will be able to trace the history of calculation machines and computer systems from Pascal to Babbage and Turing. From a more playful perspective, they will learn that magic squares have fascinated Chinese, Arab and Western mathematicians throughout the ages. Students can solve magic squares by observing constraints and using their number and operation sense, among other things.

Statistics and Probability

Polls must serve the democratic decision-making process rather than attempt to dominate it.
Harold Wilson

In order to become citizens who are able to find information, work efficiently and construct their own world-view, it is important that students develop their ability to process and analyze data. Everyday life is filled with information consisting of qualitative and quantitative data (e.g. graphs, rates, percentage, predictions, averages). This information relates to a variety of areas such as the family, population, employment, health, finance, sports, biometrics and psychometrics. Statistics and probability therefore play a central role in our society, and the tools they offer facilitate decision making in many different fields. Statistics came into its own through efforts to elaborate on observations about demographic data concerning public health, among other things. Galton developed the concepts of regression and correlation in his work on morphological measurements. Extensive data analysis has provided information that has greatly contributed to increasing human life expectancy.

The calculation of probabilities originated with games of chance. Humans have always played games that involve throwing objects (e.g. knucklebones) either to amuse themselves, to predict events or to determine the will of the gods. However, the calculation of probabilities, combinatorial analysis and mathematical expectation were developed only in the 17th century, notably by Huygens, Pascal, Fermat and the Bernoulli brothers. Students will learn that these mathematicians had varied interests, which explains why they worked on different topics. For example, Huygens's study of curves helped to improve the manufacturing of gears. Pascal studied atmospheric pressure, and the unit for atmospheric pressure bears his name.

Writers and artists (e.g. the Indians in the third century B.C.E.) have used combinatorics in their work. Raymond Queneau used it to devise and write *Cent mille milliards de poèmes* (English translation: *Hundred Thousand Billion Poems*). By selecting their own procedure, students can attempt to meet the challenge of artistic creativity by using the concepts of enumeration and combinatorics.

Students are required to make choices throughout their lives. Some broad areas of learning (i.e. *Environmental Awareness and Consumer Rights and Responsibilities, Media Literacy* and *Citizenship and Community Life*) lend themselves to the development of situations that involve aggregating individual preferences (social choice theory). A number of questions may be raised: Which is the fairest method? Which method will yield a result that best represents the will of the majority? Can we influence results? Students will discover that these methods are employed when a choice must be made, notably during elections or in establishing a ranking (e.g. for different sports such as baseball and motor racing). Students will have the opportunity to learn about such versatile researchers as Borda, Condorcet or Arrow, all of whom tried to develop methods for determining the will of a community.

There are a multitude and variety of situations that involve the concept of chance or that call for the interpretation of probabilities or an understanding of statistics. Students will be asked to determine the areas in which statistics and probability can be used to facilitate decision making, to conduct risk assessments, market studies, simulations and experiments, and to ensure quality control. Mathematical learning activities may give students an opportunity to become aware of the origin and history of the analysis of random experiments, the calculation of probabilities and the development of statistics. These activities may also spark their interest in certain mathematicians and help dispel certain myths, including those related to games of chance.

Geometry and Graphs

Inspiration is needed in geometry, just as much as in poetry.
Aleksandr Sergeyevich Pushkin

Geometric figures can be found everywhere in our everyday environment. They show up in works of art, various objects, fabrics, wallpaper, architecture, structures and trajectories, among other things. New ones are always emerging from the transformation of existing figures and the application of their properties. All cultures have different means of expression in which geometric figures feature prominently. This is notably the case in the arts. Students can develop their observational, spatial and aesthetic sense as well as analyze geometric transformations and symmetries by working with rosette patterns (e.g. hubcaps), spiral patterns (e.g. Fibonacci sequence and golden ratio), frieze patterns, tilings (tessellations) and crystals (e.g. crystallography). They can create different patterns (1D, 2D or 3D) or kaleidoscopes.

Students can identify certain properties of measuring instruments used in drawing, navigation, geodesy or astronomical observation. They can appreciate how a number of instruments (e.g. balance, odometer, Global Positioning System, compass, sextant, quadrant) used today or in the past have helped solve real-world problems. Furthermore, surveying equipment, the mirror and shadow technique, the pantograph, the proportional compasses, and Jacob's and Gerbert's staffs can help students develop their understanding of the concept of similarity. A number of mathematicians, such as Archimedes, Hero of Alexandria, Galileo and Leonardo da Vinci, designed machines, tools or measuring instruments, some of which are still used today. From a contemporary standpoint, students can become familiar with computer-assisted design software or study how computers are used to create three-dimensional representations and animations using geometric transformations, triangulation[31] and trigonometry.

Geometry and perspective are used in many fields (e.g. geography,[32] the media, computer graphics, design, engineering, architecture, photography, cinema, theatre). Different branches of geometry have been developed to address various questions and needs. In addition to Euclidean, descriptive, projective and analytic geometry, spherical and hyperbolic geometry[33] emerged in reaction to Euclid's fifth postulate. One of the most recent types of geometry to emerge is fractal geometry, which is used to model a variety of things including different natural occurrences such as atmospheric phenomena, floral patterns and geographical features. It is used in the arts and in digital imaging. Students can observe the property of self-similarity[34] in nature (e.g. ferns, trees, cauliflowers). Using recursiveness, they can attempt to create fractal objects based, for example, on the Von Koch curve or Sierpinski's triangle.

Euler laid the foundations for graph theory by trying to solve certain problems, such as the Königsberg Bridge problem. This theory is used in the different branches of mathematics [e.g. tree diagrams in probability, the representation of convex polyhedrons (planar graph)] and in various fields such as the social sciences, chemistry, biology and computer science. It can be used to solve problems related to task planning, schedule or inventory management, transportation, road or communication systems, interactions among people, electric or other types of circuits, and so on.

Different relationships can be established between famous labyrinths and graphs and between Pascal's triangle and Sierpinski's triangle. Lastly, with respect to the concept of symmetry, students can observe palindromes in numbers but also in literature (e.g. the work of Georges Perec).

31. For example, computerized imaging systems use a scatter plot to reconstruct a surface by using Delaunay's triangulation and Voronoï diagrams.
32. For example, scaled orthogonal projections provide contour lines and the Mercator projection provides a representation of the world.
33. Illustrations of this type of non-Euclidean geometry are found in certain works by M.C. Escher that are based on the work of H.S.M. Coxeter of the University of Toronto. (e.g. *Circle Limit III*)
34. Self-similarity means that a part is a reproduction of the whole.

Technical and Scientific Option

There is no branch of mathematics, however abstract, which may not some day be applied to phenomena of the real world.
Nikolay Lobachevsky

This section outlines the *concepts* and *processes* as well as the *learning processes* pertaining to the *Technical and Scientific* option. This information is outlined for each of the branches of mathematics (i.e. arithmetic and algebra, statistics and probability and geometry). It is important to approach the learning content in such a way as to show how the mathematical ideas from each branch build on one another. This section ends with a discussion of *cultural references*, which outlines activities that could help students to situate mathematical concepts in a historical and social context and to identify the needs they helped meet as well as the issues that gave rise to the development of certain processes. The study of cultural references or historical contexts allows students to appreciate the role of mathematics in daily life and in work situations as well as the contribution of numerous people to the development of this subject. Lastly, Appendix E suggests a list of principles, situations and instruments that can help students explore and make conjectures.

To help students choose the option that best suits their skills and interests, this program gave them many opportunities to explore different topics in the first year of Cycle Two. Students will continue to explore various topics in the *Technical and Scientific* option in order to better understand its focus, to use manual skills and intellectual abilities associated, among other things, with the operation of technical instruments, and to make connections between mathematics and different occupations. It is important to create learning situations that help students discover the different roles[35] played by mathematics. Some of these situations contribute to the development of the skills required in technical fields and involve contexts related to biology, physics, chemistry, the social sciences, business administration, the agrifood industry, the arts and graphic arts.

Students continue to develop their competencies in a number of ways: they compare their solutions with those of their peers; consider various points of view and exercise their critical judgment when validating a solution or a conjecture; look for the causes of a problem, for mistakes or for anomalies in solutions, algorithms or assembly drawings (e.g. architecture, landscaping) and issue recommendations with a view to taking corrective measures or making their actions more efficient. This becomes particularly meaningful in conducting case studies[36] that require the integration and application of mathematical knowledge.

These studies make it possible to examine a range of possible or probable cases in a given situation and to use proof by exhaustion. These cases raise a number of different issues relating to business or financial management, or science or technology, among other things. They pertain to operational research, the production of bids or a process of generalization based on the observation of specific cases. The variety of contexts studied may, for example, involve:

– a statistical approach in the treatment of chemical spills
– optimization that involves figures or the description of geometric loci in an architectural bid

35. See Appendix A: Aims of Mathematical Activity.
36. A case deals with one crucial aspect of reality or one factor. A case study examines a problem by analyzing certain factors associated with a given theme. Comparing cases pertaining to the same theme makes it possible to consider different crucial aspects of that theme and to make informed decisions about a particular problem. Case studies lend themselves to the use of the experimental method, among other things. They give students an opportunity to observe, work with and formulate conjectures and to verify them. Lastly, they contribute to the development and integration of mathematical and cross-curricular competencies.

– systems of inequalities in an operational research project
– the concepts of relation and function in determining priority actions aimed at stemming the spread of certain viruses
– the production of algorithms for designing or using instruments, or for making drawings or objects

Furthermore, one of the goals of this option is to make students aware of various financial considerations. Faced with situations dealing with economics, both as it pertains to business and to their personal lives, they develop an understanding of financial management and become familiar with basic processes in business administration. To that end, the different types of revenues and investments, financing, balance sheets, budgets and bids subject to various constraints can all be regarded as relevant planning and interpretation tools. The depreciation or appreciation in value of certain goods as well as gross and net revenue are also topics that help students to better understand certain social choices, the management of material goods and the financial concerns of citizens.

Actions related to *modelling, adjustment, validation* and *decision-making* processes are an important part of the learning content for this option. Students develop their critical thinking skills by validating a model and determining its limitations. They use different types of proofs and alternate between experimental reports and formal proofs. They become aware of the rigour associated with the rules and conventions involved in producing these reports or proofs. They learn to identify the principal steps in a line of reasoning, to consider different aspects or points of view and to emphasize them when communicating.

Students can carry out a number of other activities or projects that tie in with the thrust of this option. For example, they can form a committee to organize an exhibit on techniques, machines or instruments associated with mathematics. They can oversee a drawing contest whose main purpose is to illustrate the connections between functions and geometric figures. Activities may also involve special guest participants, visits to various organizations, films and the construction of scale models, among other things.

Students use both newly and previously acquired concepts and processes to deal with the situations they encounter in the last year of the cycle. For instance, even though only a small number of real functions are introduced in the last year, students are encouraged to use all the functions they have studied since the beginning of secondary school. Likewise, the concepts of distance and metric relations are an integral part of the work students do in the last year of the option. In addition, as learning content with respect to the different branches of mathematics is approached symbiotically, students' knowledge of statistics and probability comes into play in many situations, even though no new concepts are introduced in this regard.

In the last year of the cycle, students also learn to use matrices, which gives them the opportunity to expand and consolidate their knowledge of mathematics. In dealing with meaningful situations involving operations on matrices, students will become aware of how effective this register is in processing data.

In the last year of the cycle, students are required to carry out a major independent assignment. This assignment is an exploration activity in which students research and analyze mathematical information to meet specific needs or to enrich their portfolio. This activity enables students to strengthen their understanding of the concepts and processes studied in this cycle or to explore new concepts that are not compulsory at the secondary level. As it is likely to be of interest to the school community as a whole, the work carried out through this activity could be kept in the school library, posted on a Web site or published in a school magazine. Possible topics that students may be interested in exploring during this activity as well as the expected outcomes with regard to the three competencies are outlined below.

Students who choose this option are given regular opportunities to reflect on what they are doing, to explore different points of view, to act in accordance with the constraints of a situation or to adjust these constraints in order to achieve a particular result. They are encouraged to develop attitudes and abilities that are in high demand in the labour market, particularly in technical fields (whether or not they involve the use of instruments). They learn to cope with change, to deal with complex situations, to show creativity and to engage in constructive cooperation, which enables them to grow into responsible and informed citizens.

Independent assignment: exploration activity in the third year of Cycle Two

In the third year of Cycle Two, students have the opportunity to explore the cultural or work-related implications of mathematics through an activity that can be carried out at any time during the year. It is important that students be able to choose the type of work they will do in this activity and that it involve a process in which autonomy, initiative and creativity are valued. The topic they choose and the learning process they undertake must satisfy their curiosity, correspond to their interests and meet their needs. In addition to providing information on the specifics of their topic and on the concepts and processes involved, their activity report must focus on the relationship between what they did and the mathematical competencies they used. If necessary, teachers may also suggest specific activities in which they can make use of their personal expertise. The program stipulates that roughly 15 class hours are to be devoted to this activity.

The following non-exhaustive list could help students choose an activity that will allow them to discover the import of mathematics.

Instruments and techniques

- Building machines or instruments; doing research on different instruments used today or in the past: pantograph, trisector, Dürer's perspectograph, Mesolabon, curve plotter, rain gauge, oscilloscope, seismograph, etc.
- Producing a computer-assisted drawing or a two-dimensional representation of a sphere (world map)

Fields of application

Exploration of a particular field

- Activities related to architecture
- Activities related to business administration: accounting (bookkeeping, amortization, cost price); comparing the pros and cons of leasing or purchasing personal property or real estate; marketing; actuarial science; *e* number
- Analyzing the social impact of games of chance: point of view of the government (cost and revenue) and of consumers (risks, emotions, beliefs)

Exploration of a mathematical concept

- Geometric transformations (e.g. clothing manufacturers, architects, interior decorators, designers), rotation in the Cartesian coordinate system, polar coordinates (role in programming)
- Z-score, R-score, logistical correlation, different curves (e.g. contour line, distribution curve, trend line, efficiency curve, influence curve, bell curve, S-curve, regression line, logistics curve, kurtic curve)
- Numeration systems: binary systems (bar codes on products), hexadecimal and sexagesimal systems; complex numbers

Other

Historical research, book reports, concept analysis, etc.

- Trigonometric identities, factoring of second-degree trinomials, spline functions, relationship between the parameters of an equation and rotation, systems of equations with several unknowns, etc.
- Enumeration and probability in situations involving permutations, arrangements or combinations (constructing formulas); probability distribution (area under the curve); binomial distribution, normal distribution, etc.
- Spherical, hyperbolic and fractal geometry

Type of work that may be submitted

The final product can take different forms depending on the objectives involved. In all cases, however, it must include an explanation of the procedure for carrying out the activity.

- Examples of work that could be submitted: newspaper article, portfolio document, slide show, scale model, drawing, painting, etc.

The teacher could suggest various ways of presenting the results of the activity to the class, to students at another level, to community organizations, during a one-on-one interview, during an exhibit, etc.

Independent assignment: exploration activity in the third year of Cycle Two (cont.)	
Expected outcomes with respect to the competencies	**Evaluation**
In carrying out their exploration activity, students are able to recognize the actions they are taking or the strategies they are using and to associate them with the competency *Solves a situational problem* or with certain key features of this competency, depending on the activity. Students can organize their reasoning so as to show the conjectures they have made and confirmed (or refuted) and can make connections with the competency *Uses mathematical reasoning.* They identify the concepts and processes used in the activity and show their understanding of previously learned concepts and processes. Ultimately, students are able to write a report that is appropriate to the type of activity chosen. In the report, they outline their results, describe their procedure, compare their initial goals with the actual results, comment on any differences between the two and show how their actions reflect the competency *Communicates by using mathematical language.* They are able to use terminology suitable to the context of the activity and to the audience. If the project involves mathematical concepts and processes that are not prescribed in this program, their mastery is not required for the recognition of competencies.	The work in this activity may be evaluated by the teacher, the student, his or her peers, or by all these people, as appropriate. Furthermore, the teacher may draw inspiration from the evaluation criteria outlined in the program to establish suitable criteria for evaluating the work in this activity. Students must nonetheless be made aware of these criteria. Assessment of the work in this activity will be taken into account in the evaluation of one or more competencies, as the case may be.

Arithmetic and Algebra

Everything should be made as simple as possible, but not simpler.
Albert Einstein

Building on what they learned in the first year of Cycle Two, students construct and master the following concepts and processes:

<table>
<tr><th colspan="2">Understanding real numbers, algebraic expressions and dependency relationships</th></tr>
<tr><th>Concepts introduced in the second year of Cycle Two</th><th>Processes</th></tr>
<tr><td>
– Arithmetic and algebraic expressions
• Real numbers
- Radicals (nth root)
- Powers of base 2 and 10 (changing bases)
• First-degree inequality in two variables
– Relation, function and inverse
• Real function: second-degree polynomial, exponential, greatest integer (i.e. greatest integer not greater than x)
- Periodic , piecewise or step function
• Parameters
– System
• System of first-degree equations in two variables
</td><td>
– Manipulating numerical and algebraic expressions
• Writing a number using radicals or rational exponents
• Writing any number with the same base and writing a number with different bases
- Constructing and interpreting tables of values consisting of positive rational numbers written in bases 2 and 10
• Expanding and factoring expressions
- Factoring by grouping
• Graphing first-degree inequalities in two variables and checking the feasible region
• Solving exponential and second-degree equations and inequalities
– Analyzing situations
• Experimenting with concrete situations; observing, interpreting, describing and graphing them
• Modelling a situation using different registers of representation: verbal, algebraic, graphical and a table of values
- Describing the properties of a function
- Interpreting parameters
- Interpreting and graphing the inverse of second-degree, exponential and greatest integer functions
• Solving exponential and second-degree equations and inequalities
• Comparing situations and distinguishing between families of functions
• Solving systems of first-degree equations in two variables
</td></tr>
<tr><td colspan="2">Note: In situations where students must determine the approximate value of the exponent (logarithm), they use a graph, a table of values (base 2 or 10) or a calculator. In calculating this value, they manipulate expressions and convert them to the same base (base 10, for the calculator) so as to make the exponents comparable. If necessary, they use equivalences such as $a^b = c \Leftrightarrow \log_a c = b$, $\log_a c = \dfrac{\log c}{\log a}$.</td></tr>
</table>

Understanding real numbers, algebraic expressions and dependency relationships (cont.)

In analyzing different situations or experiments, students derive such information as the dependency relationship, the change, the domain and range, the intervals within which the function is increasing or decreasing, the sign, the extrema, significant values including the zero(s) and the x-intercept and y-intercept.

As regards the real functions that must be studied, students will find the rule if it is possible to express the situation using the following functions: $f(x) = ax^2$ or $f(x) = a(bx)^2$, $f(x) = ac^{bx}$, $f(x) = a[bx]$, where $c > 0$. The rule will be found using the information derived from a context, from a table of values or from a graph. The interpretation of the parameters involved is based on the context, the table of values and the graph. Changes in scale and the relationship between a change in the value of a parameter and the corresponding geometric transformation are also introduced.

For *periodic*, *piecewise* and *step* functions, graphical representations of the context are emphasized even if, in some cases, the symbolic register could be used.

The concept of two-variable inequality helps students better understand the concept of equation when they must interpret situations in a Cartesian coordinate system.

In looking for equivalent expressions, students use the distributive property of multiplication over addition or subtraction to develop their understanding of the process of factoring by grouping.

Concepts introduced in third year of Cycle Two

- Relation, function and inverse
  - Real function: sinusoidal, second-degree polynomial (general form), rational (standard form and $f(x) = \frac{ax+b}{cx+d}$ where $a, b, c, d \in \mathbb{R}$ and $cx + d \neq 0$)
  - Parameter
  - Operation on functions
- System
  - System of first-degree inequalities in two variables
  - System of equations and inequalities involving various functional models

Processes

- Manipulating algebraic expressions
  - Dividing first-degree binomials
  - Establishing the correspondence between the parameters in the standard and general forms of the equation of second-degree functions
- Analyzing situations involving real functions (functions covered in Secondary Cycle Two)
  - Experimenting with concrete situations as well as observing, interpreting, describing and representing them, using different registers of representation
  - Role of parameters in all registers of representation (context, table of values, rule and graph)
  - Solving trigonometric equations and inequalities of the first degree in one real variable that involve an expression containing a sine or a cosine
  - Finding the graphical solution for situations consisting of systems of equations and inequalities involving different functional models
- Optimizing situations that can be represented by a system of first-degree inequalities in two variables
  - Representing and interpreting the constraints and the function to be optimized
  - Determining and interpreting the feasible region (bounded or unbounded) and the vertices
  - Analyzing and interpreting an optimal solution depending on the context
  - Changing the conditions associated with the situation or changing the objective to provide a more optimal solution

Note: With respect to real functions (those studied in the options during the last two years of the cycle), the relationship between changes in the value of the parameters and geometric transformations is fully developed through the addition of parameters associated with translations. Students may nonetheless position the axes or the curve so as to make it easier to analyze a situation.

The second-degree polynomial function was introduced the year before and is now studied in standard form. Converting an expression to the general form involves developing the expression in standard form and makes it possible to establish a correspondence between the parameters. In order to switch from the general form to the standard form, students refer to the established correspondences.

Sinusoidal functions may be introduced using angles expressed in degrees.

The concepts of arcsine and arccosine are studied mainly as inverse operations involved in solving equations or inequalities. The same is true of the concepts of square root and logarithm introduced in previous years.

Learning Processes

In the *Technical and Scientific* option, the development of mathematical competencies requires regular use of algebraic concepts. The syntax and rules of algebra are introduced gradually by establishing the rigour essential to the development of the competency *Communicates by using mathematical language.* Students discover that algebraic manipulations and the concept of function are effective tools for dealing with situations. An understanding of behaviours or phenomena allows them to make decisions. Instruments and technology are useful for achieving the objectives related to the analysis of situations involving functions, and motivate students to explore technical fields.

Gradually, students develop their analytical skills and their ability to synthesize the elements of a situation by identifying related quantities and representing them in a graph or a table of values. They determine the properties of functions represented in different registers and make connections that allow them to switch between these registers. They describe different dependency relationships. They compare different models and extract information such as the type of change involved or certain critical points. In this way, they identify the families of functions covered in the program, distinguish between them and associate them with the corresponding situations. They explain why a continuous domain can be used to represent certain phenomena whose domain is discrete. In their study of functions, students come to analyze the role of the parameters of an equation and to examine how a change in the value of these parameters will affect a graph, a table of values and the data in the original context. They use their knowledge of functions to analyze statistical or experimental data, to compare or comment on results or predictions, or to make recommendations.

Analytic geometry helps students make connections between the different branches of mathematics. The geometry content to be emphasized in this approach is outlined in the *Geometry* section.

Second Year of Cycle Two

Students use the properties of exponents to understand the manipulation of numbers written in exponential or radical form. In exercising all their competencies, they use their knowledge of the connections between the different ways of writing numbers to switch from one form to another. The processes involved in changing bases to solve exponential equations or inequalities are indispensable when the initial base is not 10. It is important for students to understand these processes so that they can use technology appropriately and exercise critical judgment with respect to the results.

Students are required to ask questions when it comes to choosing an appropriate model to represent a situation. For example, does one use a step function, a first-degree function or a piecewise function to represent a per-minute long-distance telephone rate? Students learn to analyze concrete situations using periodic functions, piecewise functions or step functions. An understanding of the role of parameters allows students to describe a second-degree function algebraically, using one of the parameters (parameter *a* or *b*) under study and to establish a relationship between the resulting two equations. Furthermore, while square-root and logarithmic functions are represented graphically, the concepts of square root and logarithm are mainly taught as inverse operations involved in solving second-degree or exponential equations and inequalities related to given situations. Operations on functions can be approached intuitively, as needed. For example, the product of a function and a real number can help students understand a change in scale.

Systems of first-degree equations in two variables are solved algebraically, graphically or using a table of values. Students become familiar with a range of methods (e.g. comparison, substitution, elimination) for solving a system algebraically. They may be required to solve systems of equations when doing work in other branches of mathematics (e.g. when finding unknown measurements in geometry or unknown data values in statistics and probability).

Third Year of Cycle Two

In the third year of the cycle, students consolidate and deepen the knowledge of arithmetic and algebra that they acquire throughout Cycle Two. Irrespective of the register of representation involved, they are able to interpret the parameters of all the functions studied in Cycle Two. A number of situations modelled by a periodic function can be interpreted graphically.[37] However, only the sinusoidal model is analyzed in all the registers. Operations on functions are examined in the context of concrete situations. In addition

37. Students can deepen their understanding of the concepts of period or operations on functions by studying graphs displayed by instruments (monitors) used in health care and the media, for example.

to learning the general form and the factored form of second-degree functions, students discover that the factored form ($h(x)$) can be obtained by finding the product or the sum of two functions ($f(x)$ and $g(x)$).[38] They also learn that rational functions can be obtained by finding the quotient of two polynomial functions. In the first two years of the cycle, students began learning how to analyze situations in which the rate of change varies according to the interval in question. They continue to study this by using several functional models to describe how two variables behave in a given interval.

Students must use rigorous reasoning in order to optimize situations that involve solving systems of first-degree inequalities. When interpreting the feasible region and the vertices, with or without the help of the graph of the function to be optimized, students illustrate the reasoning they used to identify the best solution. The coordinates of a point of intersection can be determined algebraically, using matrices, or approximated based on a graph. When these coordinates are approximated, students determine whether the result is plausible and whether the degree of precision is acceptable, depending on the situation. As the case may be, students analyze the situation in order to determine the best solution, to propose changes, to suggest a new approach to solving the problem or to recommend ways of making the solution more efficient. In the process, they learn to improve the situation, while taking into account the constraints or the objective in question. Many contexts that involve solving a system of inequalities can bring different branches of mathematics into play. Students could be required to work with probability situations involving a continuous sample space. They could use their knowledge of equivalent figures in choosing an optimal solution. Systems of equations and inequalities involving various functional models may be solved by interpreting graphs with or without the help of technology. In all cases, a solution is formulated in keeping with the context and the audience.

38. e.g. product of two functions: $f(x) = a_1x + b_1$ and $g(x) = a_2x + b_2$; sum of two functions; $f(x) = ax^2$ and $g(x) = c$; $f(x) = ax^2$ and $g(x) = bx + c$; $f(x) = a_1x^2 + b_1x + c_1$ and $g(x) = a_2x^2 + b_2x + c_2$, where $a_i \neq 0$.

Statistics and Probability

The theory of probability as a mathematical discipline can and should be developed from axioms in exactly the same way as Geometry and Algebra.
Andrei Kolmogorov

Building on what they learned in the first year of Cycle Two, students construct and master the following concepts and processes:

Understanding data from random experiments and statistical reports

Concepts introduced in the second year of Cycle Two	Processes
– Conditional probability – Fairness • Odds • Mathematical expectation – One-variable distribution • Measures of dispersion: mean deviation, standard deviation – Two-variable distribution • Linear and other types of correlation - Correlation coefficient - Regression line and curves related to the functional models under study	– Interpreting probability data and making decisions concerning the data • Representing and calculating conditional probability - Distinguishing among mutually exclusive, non-mutually exclusive, independent and dependent events • Determining the *odds for* and *odds against* • Calculating and interpreting mathematical expectation • Changing the value of parameters or conditions in the situation in order to make it fair • Changing the value of parameters or conditions to optimize a gain or a loss, depending on the objectives – Analyzing statistical data dealing with one- or two-variable distributions and making decisions concerning the data • Organizing and choosing appropriate tools to present and explain the data: - Constructing and interpreting data tables for one- or two-variable distributions - Drawing graphs and scatter plots - Calculating and interpreting measures of dispersion - Interpolating and extrapolating using the functional model best suited to a situation ▪ Interpreting and describing the relationship between two variables ▪ Giving a qualitative appraisal of a correlation (strong, moderate, weak, zero) and a quantitative appraisal in the case of a linear correlation ▸ Approximating the linear correlation coefficient • Comparing distributions • Describing and critically examining a statistical study – Calculating probabilities using statistical reports, and vice versa – Anticipating (predicting, forecasting) and interpreting statistical or probability results and trends

Note: It is possible to approximate the linear correlation coefficient graphically (rectangle or ellipse method). Technology can be used to determine the value of the correlation coefficient for all models.

Learning Processes

In the *Technical and Scientific* option, the analysis of situations involving statistics and probability plays an important role in helping students develop competencies and exercise their critical judgment. In these situations, students anticipate outcomes, comment on behaviours and make decisions that they explain or justify using concepts related to statistics and probability. By using mathematics to analyze trends or beliefs that influence certain people's behaviour, students can develop a more informed opinion about actions taken and identify the probable effect of these actions. In this option, students become aware that subjective probability provides a link between statistics and probability, that a statistical study can be used to make forecasts or predictions, and that a probability does not necessarily apply to an individual just because he or she belongs to a reference class.

More specifically, in both years of this option, the concept of correlation (linear or other) will prove useful in determining the most appropriate functional model for a given situation. The equations of the models studied in the last year contain more parameters than those explored in the first year (see the *Algebra* section for more details). Probability contexts can be used to solve systems of inequalities, to extrapolate, to make recommendations or to validate conjectures.

Second Year of Cycle Two

Probability

Students continue to develop probabilistic thinking skills by studying the concept of conditional probability. Thus, in determining the probability of an event occurring given that another event has already taken place, they use the concept of dependent events, Venn diagrams or tree diagrams in order to formulate their reasoning. The situations explored should not involve the use of formulas, but enable students to use their reasoning and to represent the situation by means of different registers. By using factorial notation, they will find it easier to write out certain operations and can make efficient use of their calculator.

By exploring the concept of fairness, students learn to distinguish between the concepts of chance, odds and probability. In analyzing the rules of certain games, they can determine the *odds for* or *odds against* a player and change these rules, if necessary, to make the situation fair or more favourable to one of the players. The concept of weighted mean leads to the concept of mathematical expectation, which students use to make decisions. When analyzing situations, including games of chance, they change the parameters of the equation to make the game fair or to optimize a gain or a loss to meet certain objectives.

Statistics

The concepts and processes covered in this year of the cycle relate to the analysis and communication of information pertaining to data that students may or may not have collected themselves. They examine and present this information in order to demonstrate the reliability of a study or an experiment or to justify predictions or recommendations. In becoming familiar with these concepts and processes, students learn to investigate and discuss the how and why of a study so that they can then comment on the results. Why was the study carried out? What was it meant to prove? Is the sample representative of the entire population? What is meant by a margin of error of 3%, 19 times out of 20? Why do we say that statistics can be made to say anything? What data would be required to answer certain questions? Can a statistical study prove a conjecture? What is the relationship between the statistical approach, the experimental method in science and the mathematical modelling process?

In developing their statistical thinking skills, students strengthen their understanding of the concept of measure of dispersion in a one-variable distribution by studying the mean deviation and the standard deviation. If they understand measures of dispersion and algorithms for calculating the mean deviation and the standard deviation, they will be able to use technology intelligently to determine these values.

Students also study two-variable statistical distributions for which they construct a contingency table or a scatter plot.[39] When students study the nature of the connection between the variables in a correlation, they become aware that there is no inherent dependency relationship and that a relationship can be coincidental or linked to a third factor. This may lead them to discuss the causal or dependency relationship between the variables involved. In these cases, students estimate the equation of the regression line and the coefficient of linear correlation using an appropriate method or technology. Although different methods may be used to approximate the correlation coefficients and determine the rule of the function for other correlation models, qualitative assessment of the correlation coefficient and the use of technology should be emphasized in choosing the most appropriate functional model for a situation and in validating this choice.

39. Scatter plots representing situations involving experiments were introduced in the first year of Cycle Two in order to help students develop an understanding of dependency relationships.

Geometry

In our times, geometers are still exploring those new Wonderlands, partly for the sake of their applications to cosmology and other branches of science, but much more for the sheer joy of passing through the looking glass into a land where the familiar lines, planes, triangles, circles and spheres are seen to behave in strange but precisely determined ways.

H. S. M. Coxeter

Building on what they learned in the first year of Cycle Two, students construct and master the following concepts and processes:

<table>
<tr><th colspan="2">Spatial sense and geometric figures</th></tr>
<tr><td>Concepts introduced in the second year of Cycle Two
– Analytic geometry
• Distance between two points
• Coordinates of a point of division
• Straight line
- Equation of a straight line
- Slope
- Perpendicular and parallel lines, perpendicular bisectors
– Measurement
• Metric relations and trigonometric ratios (sine, cosine, tangent) in right triangles</td><td>Processes
– Analyzing situations
• Modelling, optimization and decision making in situations involving straight lines, the concept of distance and the point of division (Euclidean and Cartesian planes)
• Finding unknown measurements using the properties of figures and relations
- Lengths
▪ Segments in different figures
▪ Side of a triangle
▪ Altitude drawn from the right angle of a right triangle, orthogonal projection of the legs on the hypotenuse
- Areas of triangles, given the measure of an angle and the lengths of two sides or given the measures of two angles and the length of one side
- Angles of a triangle</td></tr>
<tr><td>Concepts introduced in the third year of Cycle Two
– Equivalent figures (area, volume, capacity)
– Analytic geometry
• Geometric locus and relative position
- Plane loci involving lines or circles only, and conics</td><td>Processes
– Analyzing situations
• Describing, representing and constructing geometric loci with or without the help of technology
• Defining and representing a geometric transformation using an algebraic rule or a matrix
• Modelling and optimizing situations involving the concepts of vector, distance, geometric locus, measure or equivalent figures
• Finding unknown measurements using the properties of figures and relations
- Length, area, volume or capacity of equivalent figures
- Segment, chord, arc or angle in triangles or circles</td></tr>
</table>

Spatial sense and geometric figures (cont.)	

• Standard unit circle
  - Radian, arc length
• Vector
  - Resultant, projection, operation
 	
– Measurement
• Trigonometric relations in triangles (sine and cosine laws)
• Metric relations in circles	

Note: Students also use the Euclidean plane to construct the following concepts: the distance between two points, locus, relative position and vector.

In analytic geometry, students may position the axes or the figure so as to make it easier to analyze a situation. The study of the symmetric form of the equation of a line is optional. *Vector* refers to a geometric or free vector. Operations on vectors are limited to the following: adding and subtracting vectors, multiplying a vector by a scalar and the scalar product of two vectors. These operations are performed in context.

Learning Processes

Students in the *Technical and Scientific* option will have the opportunity to continue to develop their spatial sense and to expand their network of concepts and processes with respect to geometric figures. They will develop all their mathematical competencies in exploring various geometry topics. Both an empirical and a formal (intellectual proof) approach are required to derive the properties of figures and to justify or validate the truth of various statements. When they validate conjectures, students are working with situations that involve the need to prove something, and basic theorems are the tools for doing this.

There will be many opportunities to make connections between algebra and geometry by using the concepts of geometric locus and vector, for example. Geometric transformations are also used in a number of activities. For instance, they can be used to analyze the effect of changing the values of the parameters of functions, to describe a geometric locus or to construct the image of a figure from a transformation matrix. In addition, the use of reflection makes it possible to graph the inverse of a function or to determine the minimum distance given two points and a line.

Second Year of Cycle Two

When studying the concepts of line, distance and point of division in analytic geometry, students learn to analyze situations involving the calculation of distances, perimeters or areas. Some situations may require them to show, for example, that a given line is in fact the locus of points corresponding to the perpendicular bisector of a given segment. Other situations involve determining the position of two lines in relation to each other or finding an angle of elevation from the rate of change or a given slope. They may also involve finding the distance from a point to a line or a segment. Students choose the general or standard form of the equation of a line, depending on which is the most appropriate to solve a problem.

Using the concept of similarity learned in previous years, students derive the minimum conditions required to conclude that figures are congruent or similar. They use proportional reasoning when working with metric and trigonometric relations in right triangles or in triangles that they split into right triangles.

Third Year of Cycle Two

Students broaden their network of concepts to include equivalent figures, metric relations in circles and trigonometry in triangles. The study of the standard unit circle introduces the concept of sinusoidal function, but it also provides the basis for establishing a correspondence between radians and degrees and calculating arc lengths in one of these units. The concept of vector and its geometric representation can be used to develop decision-making skills and to make connections with various fields of science. Thus, in finding the resultant, students see the relationship between vectors and a composition of translations, triangles and parallelograms. Trigonometric relations are used in situations involving the orthogonal projection of a vector.

Like the study of vectors, studying the relative position of two circles and constructing the segment representing the distance from a point to a circle or an ellipse allows students to transfer the concept of distance to new situations. The concepts of locus and relative position were introduced intuitively the previous year through the study of the analytic geometry of the line. Students continue to construct this concept through exploration and observation activities that involve finding the figure that corresponds to the description of a locus. Conversely, they describe the locus corresponding to a given figure. The emphasis is on describing a geometric locus so as to identify the necessary and sufficient conditions that make it possible to understand and use it. Thus, when students define a locus, they start by describing it in terms of the concept of distance.[40] They then use their understanding of algebraic expressions as well as familiar operations to derive equivalent expressions that correspond to that locus. Students formulate and validate conjectures relating to a locus (i.e. the possible position of a set of points that meets specific conditions). They construct loci by using properties and devising mechanisms or procedures to draw them. They modify loci or figures using geometric transformations. Constructing the concept of geometric locus therefore involves exploring several different loci, but also recognizing that a given locus can be generated in different ways.[41] Connections with the sciences and with vocational and technical training can be readily made through the study of this concept.

Situations that involve modelling and optimization give students the opportunity to use their knowledge in meaningful contexts. When designing objects in accordance with certain specifications or constraints, they will use hands-on or abstract procedures to minimize or maximize surfaces, space, or masses.

40. For example, the initial expression that directly describes the locus of a circle defined as the set of points (p_i) equidistant from a fixed point (p) could correspond to $d(p_1, p) = d(p_2, p) = d(p_3, p)$; $d(p_i, p) = r$.
41. For example, an ellipse can be described as a locus of points such that the sum of the distances of one of the points from two fixed points is constant, or as a circle that has undergone one or more geometric transformations.

Cultural References

I hope that posterity will judge me kindly, not only as to the things which I have explained, but also to those which I have intentionally omitted so as to leave to others the pleasure of discovery.
René Descartes

Mathematics has become a fundamental component of our culture, especially because of its modes of expression and representations. In this option, mathematical culture comes into play in developing competencies and mastering concepts and processes specific to the different branches of mathematics. The cultural references presented here offer an epistemological dimension that provides a window on the past, the present and the future. They provide students who wonder about the origin of a concept with avenues for learning about its evolution and current applications. They will discover that the concepts they are studying and using sometimes gave rise to a great deal of controversy before gaining acceptance and that their development often resulted from intellectual jousting between philosophers or scientists throughout history. In addition, looking at the etymology of certain terms can provide a deeper understanding of the concepts they designate.

Arithmetic and Algebra

When studying algebra, students may, for example, make a connection between the concept of periodic function and problems that involve determining lunar and solar cycles (duration of daylight) as well as the tides. These types of problems were of particular concern during the Renaissance. Operations on functions can be applied to signals processing (waves, frequencies), economics, computer science and medical imaging (nuclear resonance), among other things. Students may wonder about the possible connections between the composition of functions and fractals. In choosing an appropriate domain, whether discrete or continuous, to represent a situation, students could compare their questions with those raised by the paradoxes of Zeno of Elea.

Solving systems of equations with two unknowns may provide an opportunity to discover that over 18 centuries ago, Diophantos worked with up to ten unknowns. For example, students could attempt to determine the number of equations required to solve a system with n unknowns.

In learning about systems of inequalities, students will discover that operational research provides a number of possible strategies for dealing with real-world situations that involve decision making and the use of rigorous reasoning. Students may be interested in how operational research is applied in fields such as economics, management, agriculture, computer science and environmental science. Furthermore, linear programming makes it possible to model situations for the purpose of drawing optimal conclusions. Its industrial applications include scheduling, distribution, collective agreements, road construction (traffic flow) and clothing production. Students can learn about the relatively recent origins of this model. Leonid Kantorovich and T.J. Koopmans did research on resolving economic problems. Inspired by this research, George Dantzig developed the simplex method to solve procurement problems during the Second World War.

In developing their mathematical competencies, students may discover, among other things, that different registers of graphical representation must be used in operating a number of instruments or machines, that modelling is used to design them and that mathematical reasoning plays a role in their manufacture. On the one hand, they can take an interest in the evolution of instruments with a long history. On the other hand, they can focus on the wide variety of modern instruments (e.g. sphygmomanometer used to measure blood pressure, radar, oscillograph, multimeter) and examine their use in professional or technical occupations in the sciences. In addition, students may discover that the search for precise measurements has been a constant concern throughout history.

Statistics and Probability

In studying the concepts of fairness, odds and mathematical expectation in probability theory, students will have the opportunity to discuss game theory, which combines probability calculations and strategies and is very much used in economics. By developing their critical judgment with regard to games of chance, students become aware of the negative aspects of gambling in society.

The processing of data related to public health and the effectiveness of medication have greatly contributed to increasing human life expectancy. By discovering that statistics (recognized as a science around the 18th century) revolutionized medicine, students may come to understand the nature of the relationship between statistics and proof. The need to use statistics to demonstrate a possibility (e.g. to show that a vaccine works) led to the calculation of the margin of error and to the establishment of decision intervals, which have proven to be essential tools in many fields. Meteorology definitely lends itself to conjecture, since it is an always-evolving science involving a variety of measuring instruments and characterized by the uncertainty of forecasts based on the combination of different types of data.

Geometry

To help students organize their knowledge and structure their procedures, it would be useful to have them learn about those who went through the same process a long time ago. For example, we owe Euclidean geometry to Euclid, a Greek mathematician who developed an organized body of geometric principles. In using deductive reasoning, students will learn how to construct proofs. In the process, they may discover that for Aristotle, deductive reasoning was the best way to acquire knowledge and that for Galileo and Descartes, deductive reasoning made it possible to produce mathematical explanations of physical phenomena.

For example, students can use their spatial sense when describing anomalies in "so-called" impossible figures (e.g. Escher) or in assembly drawings that do not always correctly represent the required object. In developing their measurement sense, they will learn to appreciate how a number of instruments (e.g. odometer, Global Positioning System, compass, sextant, quadrant) used today or in the past have helped solve many problems. Furthermore, surveying equipment, navigation and astronomical instruments, the mirror and shadow technique, the pantograph, the proportional compasses, and Jacob's and Gerbert's staffs can help students to develop an understanding of the concept of similarity or to make connections with the field of science. In computer science, they may discover, among other things, that visual on-screen representation involves trigonometry and that animation in the development of video games involves geometric transformations.

Students are introduced to analytic geometry during this cycle. The combination of loci (geometry) and equations (algebra) makes it easier to compare mathematical objects. In learning to express the properties of geometric figures algebraically, students could be asked to compare their work with Descartes' effort to lay the foundations for a synthesis that would prove to be an invaluable achievement in the history of mathematics. Given that astronomy is a science that combines algebra and trigonometry and that robotics, mechanics, automotive production and 3-D description involve combining sets of loci with algebra, all these fields could be incorporated into the learning process because they are likely to pique the interest and curiosity of students. When looking for loci related to conics, students could be informed about Kepler's work on modelling the elliptical orbit of the planets and about the fact that Kepler devised an equation to describe this. Newton later proved that this equation was inaccurate, since the attraction between the planets causes them to deviate slightly from a perfectly elliptical path.

In looking for efficient solutions, students master a specific register of representation (i.e. matrices) used in many fields that could open up career options previously unknown to them. Examples of this include situations that involve using geometric transformations to solve computer graphics problems. Furthermore, in situations that involve optimizing the shape of a figure or its measurements (length, area, volume or capacity) according to certain specifications, students could use the concept of equivalent figures and Cavalieri's principle to develop a line of deductive reasoning about cross-sections of solids.

In addition, students may find spherical and hyperbolic geometry interesting, even though they are not compulsory in the program. Fractal geometry, used in the arts and digital imaging among other things, involves the fascinating task of modelling atmospheric phenomena, floral patterns or geographic features.

In developing their competency to communicate and in formulating different points of view or opinions, students will discover that mathematics and philosophy have long been linked and that philosophizing about mathematics can be both fun and instructive.

Science Option

The most vitally characteristic fact about mathematics is, in my opinion, its quite peculiar relationship to the natural sciences, or more generally, to any science which interprets experience on a higher than purely descriptive level.
John von Neumann

This section outlines the *concepts* and *processes* as well as the *learning processes* pertaining to the *Science* option. This information is outlined for each of the branches of mathematics (i.e. arithmetic and algebra, statistics and probability and geometry). This section ends with a discussion of *cultural references,* which outlines ways of situating the mathematical learning content in a historical and social context and relating it to the sciences. Lastly, Appendix E suggests avenues of exploration that can help students make conjectures.

In the *Science* option, students continue to develop their competencies, use and expand their knowledge, and become familiar with new networks of concepts and processes. Their capacity for abstract thinking enables them to make a variety of connections among the different branches of mathematics, and notably between algebra and geometry. They make more formal use of symbols, rules and conventions in their work and are required to construct proofs.

This option emphasizes the modelling process. By learning to devise a mathematical model to represent a situation, students develop the ability to work with various types of dependency relationships, geometric figures and statistical processes. Students analyze a situation, a phenomenon or a behaviour and notice related patterns or trends. They interpolate, extrapolate and generalize elements. These activities may involve simulations or making connections between statistical and algebraic concepts. In this way, students discover how useful mathematics can be in interpreting reality and in making generalizations, predictions and decisions.

Students encounter situations that require them to use their knowledge of mathematics and other subject areas. They work with purely mathematical contexts, while continuing to deal with concrete situations, particularly of a scientific nature.

The learning situations should preferably be related to the sciences, since they enable students to develop methods used in scientific research and investigation. The variety of situations that may be studied can focus on such things as:

– contexts involving biology or economics and that can be represented by exponential functions (e.g. cell multiplication, epidemics or the study of different financing rates and terms)
– cyclical occurrences that can be represented by periodic functions (e.g. tides, the seasons, physiological data, mechanisms that generate movement, the changes in a person's position or a pendulum in motion)
– demographic or biological contexts involving estimates, simulations and statistics (e.g. determining the quantity of fish in a body of water over a given period)
– contexts associated with physics and involving the concepts of slope, distance, speed and vector. For example:
  - analyzing situations involving successive displacements, forces, speeds or velocities
  - determining speed or velocity by analyzing the slope of a graph representing distance as a function of time
  - finding the distance travelled by calculating the area under the curve
  - comparing the slope of a line that intersects a curve with the average speed of a moving object in a given interval

Other mathematical concepts may come into play and pertain to such fields of activity as surveying, topography, geodesy, biology, biometrics, optics, mechanics, electricity, chemistry, meteorology or computer science.

In addition, students can be assigned science-related mathematical activities in order to emphasize the scientific aspect of this option as well as the contribution of mathematics to society. Among other things, they could organize exhibits, interview a physicist or a mathematician, or visit pharmaceutical, meteorological or robotic centres. Students could also create a mathematics tutoring committee to help younger students with the subject. Such activities enable students to develop a positive attitude toward mathematics, to better understand different concepts that are based on mathematics and to find new ways of being actively involved in their own education.

The learning process can also be enhanced by technology. By piquing their curiosity, the use of technology encourages students to reason and formulate various conjectures. The visual nature of these tools helps them form a mental image of the situations they encounter. Some types of dynamic software allow them to observe what happens when the values of certain parameters are changed. These different tools give them an opportunity to generate various conjectures that in turn give rise to other questions and gradually lead to a process of exploration that will help them construct a proof.

In the last year of Cycle Two, students are required to carry out a major independent assignment involving a detailed investigation activity that gives them an opportunity to use their competencies and to work with the concepts and processes they have learned. Furthermore, in order to better meet the needs of certain students, this thematic activity may involve learning content that is not prescribed in the program. A list of topics has been provided below to help students define the focus of their detailed investigation activity.

Students have many opportunities to incorporate the broad areas of learning, refine their work methods, explore research procedures, discuss science and develop mathematical and cross-curricular competencies in meaningful situations. As the main architects of their own education, students carry out these activities autonomously. This option offers them an intellectual education that prepares them to function effectively in a changing world.

Independent assignment: detailed investigation activity in the third year of Cycle Two

Students in the *Science* option have the opportunity to expand their knowledge of mathematics by calling upon their creativity, critical judgment and ability to use information in carrying out a detailed investigation activity. If necessary, they may start work on this activity at the beginning of the school year.

Through this activity, students will delve further into certain mathematical concepts, learn about the use of mathematics and its contribution to various fields of activity, become aware of the mathematical skills involved in performing various tasks and have the chance to demonstrate perseverance and autonomy. This activity is therefore aimed at encouraging students to use their mathematical competencies in a context that is meaningful to them. The program stipulates that roughly 15 class hours are to be devoted to this activity.

Although it is not exhaustive, the following list may help students select a topic that will guide them in choosing a career and allow them to achieve significant learning. Some of these topics lend themselves to the use of business administration or applied science concepts.

- Computer-aided drawing, polar coordinates and role in programming, vectors in space
- Work related to architecture, astronomy and different types of geometry (e.g. Euclidean, spherical)
- Metric relations in a circle
- History of trigonometry: trigonometric ratios and the standard unit circle (secant, cosecant, cotangent)
- Complex numbers, fractals and complex iterated functions
- Solving equations with several unknowns, matrices
- Conditional probability, mathematical expectation
- Areas of regions bounded by a conic
- Surfaces of revolution: paraboloid, ellipsoid and hyperboloid
- Enumeration and probabilities in situations involving permutations, arrangements or combinations
  - Construction and use of formulas
- Probability distribution: binomial distribution, normal distribution, Poisson distribution, geometric law, hypergeometric law
  - Graphical representation of a distribution; area under the curve; use of a probability distribution table; hypothesis testing (H0, H1), parameter estimation, confidence level, confidence interval
- Financial mathematics: accounting methods
- Geometric series: annuities, compound interest and solving problems involving financial situations

Type of work that may be submitted

The final product can take different forms depending on the objectives involved. In all cases, however, it must include an explanation of the procedure for carrying out the activity.

- Research report
- Summary poster
- Other

The teacher may suggest different forums for presenting the results of the activity: in-class presentation, an exhibit or a one-on-one interview.

Independent assignment: detailed investigation activity in the third year of Cycle Two (cont.)

Expected outcomes with respect to competencies

In carrying out their detailed investigation activity, students solve a situational problem by bringing all the key features of the competency into play. They are provided with a good opportunity to explore their interests and use their aptitudes. The activity may involve mathematical concepts that are not prescribed in the program and that students analyze systematically by decoding and modelling the elements that can be processed mathematically. Drawing on the many connections between the various branches of mathematics can also enhance the activity, while enabling students to delve further into the compulsory concepts and processes.

Students use their mathematical reasoning especially by drawing upon the networks of concepts and processes they have constructed. They build on their knowledge by organizing new mathematical objects. They formulate conjectures, propose justifications and make connections between different concepts throughout this activity. Their detailed investigation procedure is organized and clearly articulated.

Students communicate by using mathematical language in developing the activity (reading, interpreting, synthesizing) and presenting the final product (report, presentation, poster, article, exhibit). From the range of semiotic representations at their disposal, they choose those that are the most useful for adapting their message to the reactions and questions of their audience.

Evaluation

The work in this activity may be evaluated by the teacher, the student, his or her peers, or by all these people. Furthermore, the teacher may draw inspiration from the evaluation criteria outlined in the program to establish suitable criteria for evaluating the work in this activity. Students must nonetheless be made aware of these criteria. Assessment of the work in this activity will be taken into account in the evaluation of one or more competencies, as the case may be.

Arithmetic and Algebra

Mathematics is the queen of sciences and number theory is the queen of mathematics.
Friedrich Gauss

Building on what they learned in the first year of Cycle Two, students construct and master the following concepts and processes:

<table>
<tr><th colspan="2">Understanding real numbers, algebraic expressions and dependency relationships</th></tr>
<tr><th>Concepts introduced in the second year of Cycle Two</th><th>Processes</th></tr>
<tr><td>
– Algebraic expression
• Algebraic identity (of the second degree)
• Second-degree equation and inequality in one variable
– Real function
• Step function
- Greatest-integer function (greatest integer not greater than x)
• Second-degree polynomial function
• Parameter
– System
• System of first-degree equations in two variables
• System composed of a first-degree equation and a second-degree equation in two variables
</td><td>
– Manipulating algebraic expressions
• Multiplying algebraic expressions
• Dividing polynomials (with or without a remainder)
• Factoring polynomials
• Expanding, simplifying or substituting expressions, using significant algebraic identities
• Solving first- and second-degree equations and inequalities in one or two variables, depending on the context: algebraically or graphically
- Validating and interpreting the solution
– Analyzing situations
• Observing, interpreting and describing different situations
• Representing a situation involving a real function: verbally, algebraically, graphically and using a table of values
- Observing patterns
- Describing the properties of a function
- Interpreting parameters
- Finding the rule of a real function
- Switching from one form to another in writing second-degree polynomial functions: general form, standard form and factored form
- Interpolation and extrapolation
• Interpreting results
• Comparing situations
- Solving systems of equations using a table of values, a graph or algebra
- Interpreting the solution(s), depending on the context
- Choosing an advantageous solution
</td></tr>
</table>

Understanding real numbers, algebraic expressions and dependency relationships (cont.)

Note: In analyzing different situations, students derive such information as the dependency relationship, the change, the domain and range, the intervals within which the function is increasing or decreasing, the sign, the extrema, significant values including the zero(s) and the *x*-intercept and *y*-intercept.

The rule of a second-degree polynomial function is found using the vertex and another point or the zeros and another point.

The concept of two-variable inequality helps students better understand the concept of equation when they must interpret situations in a Cartesian coordinate system.

Concepts introduced in the third year of Cycle Two	Processes
– Arithmetic and algebraic expressions • Real numbers: absolute value, radicals, exponents and logarithms	– Manipulating arithmetic and algebraic expressions, using the properties of radicals, exponents, logarithms and absolute values • Solving equations and inequalities in one variable: absolute value, square root, rational, exponential, logarithmic, trigonometric
– Relation, function and inverse • Real function: absolute value, square root, rational, exponential, logarithmic, sinusoidal, tangent - Piecewise • Operations on functions	– Analyzing situations • Observing, interpreting and describing different situations - Modelling situations and drawing a scatter plot ▪ Determining the type of dependency relationship, interpolating and extrapolating using the line of best fit, with or without the help of technology • Representing a situation involving a real function: verbally, algebraically, graphically and using a table of values - Finding the rule of a function or its inverse, depending on the context - Describing the properties of a function
– System • System of first-degree inequalities in two variables • System of second-degree equations (with respect to conics)	– Optimizing a situation, taking into account different constraints • Representing a situation, using a system of equations or inequalities • Solving a system of equations or inequalities: algebraically or graphically - Representing and interpreting the solution set • Choosing one or more optimal solutions • Analyzing and interpreting the solution(s), depending on the context

Note: In consolidating their understanding of the properties of integral and rational exponents, students

– expand their knowledge of the laws of exponents: $a^m \times a^n = a^{m+n}$, $a^m \div a^n = a^{m-n}$, $(ab)^m = a^m b^m$, $\left(\frac{a}{b}\right) = \left(\frac{a^m}{b^m}\right)$

– deduce the different properties of radicals: $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$, $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$, $\left(\sqrt{a}\right)^2 = a$, $\frac{a}{\sqrt{b}} = \frac{a\sqrt{b}}{b}$, $\frac{1}{\sqrt{a}+\sqrt{b}} = \frac{\sqrt{a}-\sqrt{b}}{a-b}$

– deduce the following equivalences, using the relationships between exponents and logarithms: $a^b = c \Leftrightarrow \log_a c = b$, $\log_a c^n = n\log_a c$, $\log_a c = \frac{\ln c}{\ln a} = \frac{\log c}{\log a}$

The study of exponential and logarithmic functions should focus on bases 2, 10 and e.

Step functions are considered to be a specific type of piecewise function.

The concept of inverse is mainly associated with rational, exponential, logarithmic and square root functions.

Learning Processes

Starting from concrete situations, students in this option develop models, hone their capacity for abstract thinking and transfer what they have learned to new situations, concrete or otherwise. The ability to work with algebraic expressions will help them become familiar with the modelling process, which is intrinsic to situational analysis. This ability can also be used to solve systems of equations and prove the identity of algebraic expressions. In exercising their competencies to solve equations or inequalities, students use geometric concepts or simplify different algebraic expressions.

Second Year of Cycle Two

In analyzing certain situations (including experiments) using real functions such as polynomial functions of degree less than 3 or step functions, students may be required to use statistical tools (scatter plots and linear correlation) and to examine and interpret such things as economic contexts or physical phenomena related to trajectories. Students determine whether a situation is represented by a constant function, a first-degree function, a greatest-integer function or a second-degree function. They are able to interpolate or extrapolate from the rule of a function or its graph. In addition, step functions give them the opportunity to further develop their understanding of real numbers and their reasoning abilities, especially when they represent and compare *greatest-integer, truncation,* and *round* functions as well as the *fractional part* function.

In order to determine the parameters of the functions to be studied, students will normally be required to write the standard form of the equation: $f(x) = a[b(x-h)] + k$ and $f(x) = a(x-h)^2 + k$. To make the parameters meaningful, students analyze the role of these parameters in the rule of the function, their effect on graphs (transformation of the initial function) as well as their relationship with the information provided in the situation. Observations and manipulations can be carried out with or without technological tools, depending on the educational goals involved. Technology makes it possible to pinpoint the relevant model more quickly and to emphasize analysis and justification rather than algebraic operations.

Third Year of Cycle Two

Students explore a number of situations that can be represented by a periodic function. The standard unit circle provides a context for analyzing the specific case of trigonometric functions. This circle enables students to visualize the periodic nature of trigonometric functions and trigonometric lines, to derive properties and to prove certain identities. The use of piecewise functions makes it possible to analyze a variety of situations, such as remuneration during and after regular working hours (time and a half, double time). A rule is determined and defined for each interval of the domain. The concept of continuity then comes into play and can be used to interpret the variation in the rate of change.

Operations on functions are studied in practical contexts that, for example, involve calculating the total income tax payable (addition) or sales taxes (composition). The study of these operations must not be an end in itself, but should be part of the process of analyzing situations and developing corresponding models.

The concepts of infinity and continuity allow students to understand the asymptotes of functions and vice versa. The definition of the concept of limit is introduced intuitively (without referring to symbols) in order to clarify certain situations. The study of rational, tangent, exponential or logarithmic functions can also give rise to a discussion of these concepts.

Furthermore, when looking for the optimal solution in a situation involving a set of constraints, students use linear programming. They use a system of inequalities in two variables to represent the different constraints and an equation to define the function to be optimized. They draw a graph of the situation and study the resulting polygon of constraints. To choose an optimal solution, they determine the coordinates of the vertices graphically or algebraically. They justify their choice and interpret it in light of the context.

Statistics and Probability

It is remarkable that a science which began with the consideration of games of chance should have become the most important object of human knowledge.
Pierre-Simon de Laplace

Building on what they learned in the first year of Cycle Two, students construct and master the following concepts and processes:

Understanding data from statistical reports	
Concepts introduced in the second year of Cycle Two	**Processes**
– Two-variable distribution • Linear correlation - Correlation coefficient - Regression line	– Organizing and analyzing a two-variable distribution • Drawing a scatter plot • Assessing a correlation qualitatively and quantitatively - Representing and determining the equation of the regression line - Approximating the linear correlation coefficient with or without the help of technology - Interpreting the linear correlation coefficient

Learning Processes

In developing statistical thinking skills, students in the *Science* option apply the concept of dispersion to the study of two-variable statistical distributions, which they represent using a contingency table or a scatter plot. Analysis of the scatter plot makes it possible to describe and characterize the correlation in a qualitative fashion (perfect, strong, weak, zero, positive, negative). They become aware that handling or measurement errors affect the results of experiments. Thus, the resulting graphs are not always "perfect" curves. By analyzing various situations or experiments related to the development of their competencies, students become aware that a mathematical model, such as a function, can be associated with a scatter plot.

Second Year of Cycle Two

Students will be able to define the concept of linear correlation and use it to verify the strength of the relationship between two quantities. They approximate the correlation coefficient using a graphical method or technology. Using different methods, they determine the rule that corresponds to the line of best fit in order to describe the relationship observed and to interpolate or to extrapolate. Students become aware of the fact that a strong correlation does not necessarily mean that a causal link exists. The relationship between two values may be coincidental, explained by a third factor, or causal.

Students use their mathematical reasoning when extrapolating from a statistical analysis. For example, they can use statistical data about certain athletic performances and predict various outcomes.

Geometry

You, who wish to study great and wonderful things, who wonder about the movement of the stars, must read these theorems about triangles. Knowing these ideas will open the door to all of astronomy and to certain geometric problems.
Regiomontanus

Building on what they learned in the first year of Cycle Two, students construct and master the following concepts and processes:

Spatial sense and geometric figures	
Concepts introduced in the second year of Cycle Two	**Processes**
– Equivalent figures – Analytic geometry • Line and distance between two points – Measurement • Metric and trigonometric relations in triangles: sine, cosine, tangent; sine and cosine laws	– Analyzing situations • Finding unknown measurements, using the concept of distance and the properties of congruent, similar or equivalent figures - Angles of triangles or of figures that can be split into triangles - Lengths ▪ Segments resulting from an isometry or a similarity transformation ▪ Side of a triangle ▪ Altitude to the hypotenuse of a right triangle and orthogonal projection of the legs on the hypotenuse - Areas and volumes of figures
Concepts introduced in the third year of Cycle Two	**Processes**
– Analytic geometry • Standard unit circle - Trigonometric identity • Vector • Conic - Parabola - Circle, ellipse and hyperbola centred at the origin	– Manipulating trigonometric expressions • Expanding, simplifying or substituting expressions using trigonometric identities – Analyzing situations involving the concepts of congruence, similarity, geometric transformation, conic and vector • Finding unknown measurements • Operations on vectors - Adding and subtracting vectors - Multiplying a vector by a scalar - Scalar product • Describing a situation by means of a figure, a vector or a rule • Describing the elements of a conic: radius, axes, directrix, vertices, foci, asymptotes, regions • Finding the rule (standard form) of a conic and of its internal and external region • Determining the coordinates of points of intersection between a line and a conic or between a parabola and another conic

Note: Students prove the identity of simple trigonometric expressions algebraically or geometrically, using definitions (tangent, secant, cosecant, cotangent), Pythagorean identities, properties of periodicity and symmetry as well as formulas for finding the sum or difference of two angles.

Hero's formula could be introduced in order to calculate the area of a scalene triangle.

Vector refers to a geometric or free vector.

Learning Processes

Ideally, the properties studied in this option should emerge as conclusions derived from the exploratory activities carried out by the students. These properties help them to justify their procedures when they exercise their mathematical competencies. They deduce certain properties through structured reasoning that is based on accepted definitions, relations or properties and use an appropriately rigorous approach.

Second Year of Cycle Two

In analytic geometry, the calculation of the distance between two points comes into play in analyzing situations involving functions or geometric figures. Straight lines are studied in conjunction with systems of first-degree equations. In this case, special attention should be paid to the different forms of the equation of a line (standard, general and symmetric) and to the relative positions of lines (intersecting at one point, parallel [non-intersecting], coincident, perpendicular).

Through exploration and deduction, students determine the minimum conditions required to conclude that triangles are congruent or similar. They use these properties to find unknown measurements. By using proportional and geometric reasoning as well as their knowledge of similar triangles, they discover the different metric relations in right triangles.

The standard unit circle is used to introduce concepts related to the trigonometry of right triangles. Together with the Pythagorean theorem, the determination of the coordinates of certain points (serving to form angles of 30°, 45° and 60°) makes it possible to establish certain significant trigonometric values. The concepts of dilatation and similarity of triangles complete the study of trigonometric relations. This approach allows students to explain the existence of negative values in trigonometric ratios or the existence of ratios of the same value for two different angles.Various avenues of exploration will stimulate the students' curiosity. The basic idea is to get students to make conjectures while exploring families of figures and to thereby exercise their reasoning in a geometric context. For example, they may discover that for any type of triangle, the length of the segment joining the midpoints of two sides of a triangle remains constant despite the fact that the vertex shared by the resulting triangles is shifted parallel to the base. Several other avenues of exploration enabling students to exercise their ability to make conjectures are given in Appendix E.

Third Year of Cycle Two

Students consolidate their knowledge of the relationships between geometry and algebra by using trigonometric identities and studying conics, among other things. With respect to trigonometry, they use their understanding of equivalence relations and their ability to work with algebraic expressions to prove identities involving trigonometric expressions and to solve trigonometric equations. In studying conics, students discover other applications, notably with regard to telecommunication systems. This also enables them to deepen their understanding of and to apply various concepts and processes related to optimization, algebraic calculations or relations. Students examine conics based on a cross-section of a cone or through various hands-on activities (folding, play of light and shadows, construction). They observe patterns and attempt to define the different conics. They determine the equations associated with them and describe each region by using inequalities. They find the coordinates of points of intersection and different measurements using algebra and changing variables when necessary.

In examining the concept of vector, students build on what they learned about linearity in the previous cycle. Vectors make it possible to take a new approach to certain situations involving geometry and can be related to different concepts such as proportionality, linear functions, first-degree equations and geometric transformations associated with motion. Students can then compare the properties of real numbers with those of vectors. When performing vector operations, they use the Chasles relation, among other things. Depending on the situations involved, students can also work with different linear combinations or determine the coordinates of a point of division using the product of a vector and a scalar. Vectors are studied in both the Euclidean and Cartesian plane.

Technology could enable students to visualize the intersection of conics or to examine, for instance, curves of the form $y^2 = x^3 + ax + b$, which possess very special arithmetic properties that can be used in cryptography.

Cultural References

But there is another reason for the high repute of mathematics: it is mathematics that offers the exact natural sciences a certain measure of security which, without mathematics, they could not attain.
Albert Einstein

The cultural references discussed here show that mathematics is part of the heritage of humanity and that this discipline has evolved over time. These references constitute possible ways of integrating the cultural dimension into the *Science* option. There are many connections between mathematics and science, and these disciplines have been closely related through the ages. Mathematical advances have certainly made it possible to solve countless problems and to meet various social needs; these advances have also often preceded concrete applications and their use in science and other fields. The great thinkers of past were usually conversant in several disciplines: philosophy, mathematics, science, the arts, and so on. In contrast, today's researchers must become specialists, but the disciplines are still connected. The following pages describe some of the historical and contemporary contributions that the different branches of mathematics have made to civilization.

Arithmetic and Algebra

Various fields give students the opportunity to observe and manipulate number expressed in different types of notation. For example, exponential notation is widely used in computer science, in particular to represent and locate the billions of Internet addresses. This representation is essential in cryptography to ensure secure transmissions and transactions. Exponential notation is also used in demography and geography to illustrate different growth models, including geometric progression.

With regard to economics and the consumption of goods and services, activities involving percentages, exponents and large numbers enable students to appreciate how mathematics is used in their everyday lives. For example, they can analyze the composition of bar codes on packaging. Based on the binary system and decoded by an optical scanner, this virtually indispensable code has revolutionized industrial activity. It makes it possible to process purchases more rapidly and to automatically update inventories.

Mathematics is responsible for the existence and effectiveness of a growing number of objects, tools and techniques used on a daily basis. For example, recent developments in weather forecasting, digital image processing, data fusion related to aerial and space-based surveillance, the control of rail transport, the optimization of cellular telephone networks, hydroelectric management of a power station or region all involve mathematical modelling.

Algebra contributes to the modelling process and to the development of abstract thinking. For example, students can compare their own and their classmates' understanding of abstract concepts such as infinity and trace the evolution of this concept throughout history. The ancient Greeks had difficulty differentiating between notions of the infinite and the continuous and notions of the finite and discrete. The paradoxes created by Zeno of Elea can serve as the starting point for a discussion on this topic. For example, the students could examine the paradox of the arrow, which evokes the concept of instantaneous velocity, by raising the following question: What value must be assigned to the ratio between the distance covered and an interval of time ($\Delta x/\Delta t$) as the latter quantity (Δt) becomes very small? This question is problematic because the value of the denominator eventually reaches zero. With the introduction of infinitesimal methods in the 17th century, particularly through the work of Leibniz and Newton, the concept of limit now makes it possible to solve this kind of problem, since instantaneous velocity is the limit of this ratio when Δt approaches zero.

Equations make it possible to perform calculations that will lead to informed decision making. They are used in such varied fields as operational research, communications, simulation games, the design of amusement park rides (calculation of the moment of inertia, the braking distance or centripetal acceleration), etc.

For example, certain formulas can be used to calculate the velocity of propagation of electromagnetic waves or to determine the area of a satellite's panels. In their work, air traffic controllers must provide a block of airspace for every aircraft that flies into the area for which they are responsible. They use the relation $d = vt$, which allows them to call for a change in the speed of an airplane or to adjust the air corridor to ensure that all aircraft will be able to land without a problem. Equations such as those pertaining to conics make it possible to describe different phenomena associated with trajectories (e.g. falling bodies, gravitational orbits).

Operational research provides tools for improving efficiency in such fields as agriculture, the environment, engineering, computer science, logistics, medicine, telecommunications, transportation and the economy. Optimization strategies are used in public and private utility services, for instance in scheduling work shifts (day, evening, night) or in organizing transportation. In emergency transportation logistics, vehicles are positioned in strategic spots to minimize the time it takes to respond to an emergency call, which thereby ensures that a given territory receives optimal service. Students will realize the importance of programming and optimization processes in developing solutions that meet needs and comply with constraints.

Statistics and Probability

It is important that students be able to use statistical tools effectively and that they be able to use probability models to interpret random events as well as the results of experiments and surveys. Statistics and probability are increasingly part of our daily lives. They are used in a variety of fields such as economics, insurance, biology and medicine. In physics, probability theory must be used to model complex phenomena (e.g. theory of gases, Brownian motion).

In experimental research, probability concepts are used to develop models and simulations, whereas statistical processes are used to collect and represent data, interpret results, take account of errors (measurement, sampling) and make predictions. Mathematical expectation, correlation, conditional probability, the law of large numbers and the normal distribution are concepts that students can use in carrying out activities. For example, correlation makes it possible to define models and make connections between variables.

Statistical and probability theory was developed as a result of the extensive work of scientists such as Pascal, Fermat, Huygens, Galton, Bernoulli, Gauss, Laplace, Poisson, De Moivre, Quételet and Komolgorov. Students will be able to appreciate the importance and contribution of these mathematical concepts and processes in the context of various scientific activities.

Geometry

Geometry has a rich history. The Greek thinkers were geometers first and foremost. They worked on abstract objects and organized geometry deductively. Students learning how to deal with abstraction and how to apply the principles of deduction may be interested in learning about the major contributions of these mathematicians and how their ideas evolved.

Thales of Miletus is recognized as the founder of Greek geometry. Euclid's important contributions cannot be overlooked either; his treatise entitled *Elements* has influenced many generations of philosophers and mathematicians. In their wake, other thinkers such as Apollonius, Archimedes, Ptolemy, Hero and Diophantos built on the knowledge of geometry and trigonometry prevalent in the classical period by making connections with other disciplines such as mechanics and astronomy.

For centuries, trigonometry was associated almost exclusively with astronomy. Hipparcus and Ptolemy established tables of numbers to make it easier to perform various calculations. These tables consisted of the lengths of segments (chords and half-chords) for different angles within a circle. The Arabs further developed these tables and used them for religious purposes, among other things. Only by the end of the 16th century was trigonometry used in fields other than astronomy to solve problems related to measurement and surveying, among other things. It was during this time that sine, cosine and tangent were defined in terms of the ratio between the lengths of the sides of a right triangle. By studying the history of trigonometry in greater detail, students will see that many astronomers and mathematicians from different cultures have contributed to its development. They will discover that trigonometry made it possible to address problems related to geocentrism or various religious considerations. Later, it was used to facilitate certain calculations in physics through the definition of a new unit of measure: the radian.

To help students use their mathematical reasoning, the teacher could discuss Plato and Aristotle, both of whom have profoundly influenced Western thought. The first principles of geometry emerged during Plato's time. Aristotle, one of the founders of logic, codified the laws of reasoning and used them as an instrument of thought that has its own rules and standards. In ancient Greece, mathematics and philosophy were closely linked. To make students aware of the major questions associated with the beginnings of mathematics, the teacher could present them with situations similar to classical problems such as the duplication of the cube (a legendary problem according to Eutocius, where Apollo is alleged to have ordered that one of the altars in his sanctuary be doubled in size), the trisection of an angle (dividing a given angle into three parts) and squaring the circle (constructing a circle and a square of the same area, with an ungraduated ruler and a compass). From the outset, geometry problems played a key role in the history and development of mathematics.

René Descartes, the father of analytic geometry, also conducted research on optics and meteorology. He thought that the world was a reflection of geometry, that it conformed to simple mathematical laws and that everything could be expressed mathematically. Like Descartes, Galileo believed that science should be modelled on mathematics and based on axioms and that deductive reasoning should be used to identify new properties and formulate new propositions. Aristotle expressed the same opinion some 20 centuries earlier, in *Organon*, when he wrote that "knowledge comes only from demonstration." The great philosopher distinguished between the definitions of axiom, hypothesis and so on. Thus, knowledge did not simply mean having information, as Plato claimed; it involved being able to produce an explanation in accordance with certain rules of logic.

The concepts of volume and solid are used in many different fields, including those that involve planning (e.g. transportation, storage and inventory management). When they study equivalent figures, students could be encouraged to develop their ability to think abstractly by exploring Cavalieri's Principle, from which concrete applications may be derived. The study of symmetry and shapes can also be applied in chemistry in order to understand the structure of molecules and crystals. Architects also use geometric concepts in developing plans.

Vectors provide students with another way of understanding reality. They are used to represent forces or trajectories, among other things, and are applied to everything that involves motion (e.g. aviation). Air traffic controllers use vectors to provide a three-dimensional representation of the movement of airplanes passing through the area for which they are responsible. Engineers who design structures or aircraft such as helicopters constantly use them to represent the forces that interact on these objects.

Students will come to see that geometry has evolved over the centuries to meet human needs and that it is important to be aware of some of the developments that occurred in the past in order to be able to interpret present-day reality. Today, everything is digitized: images, sound, photographs, communications, and so on. Digital compression has made it possible to transmit these types of information more rapidly. What does the future hold in store in this area?

APPENDIX A – AIMS OF MATHEMATICAL ACTIVITY

Individuals form mental images, formulate judgments and develop their aesthetic sense through reflective interaction with their environment. When interpreting real-life situations, individuals learn to assess the relevance of the information they receive, to deal with various related influences and to feel confident about the actions they take.

Mathematics is an ideal tool for interpreting real-life situations. It enables us to understand and describe the physical environment by providing us with resources related to spatial sense such as representation, position and movement, the order of magnitude, the location of objects, scales and measurement. The analysis of information, behaviours and phenomena becomes meaningful when data is processed by means of observation, modelling, correlation, dependency relationships, graphs, statistics and probability as well as proportional reasoning.

TO INTERPRET REALITY

in order to …
analyze and understand the world and develop an informed and critical view of it.

Individuals can anticipate the outcome of their approach effectively by relying on their intuition, experience, and ability to compare and generalize situations. The ability to anticipate makes it possible to plan, visualize impacts and effects, and determine the actions to be taken throughout the process of dealing with a given situation.

Mathematics contributes to the development of this aptitude by teaching individuals how to approximate the result of operations, to enlarge or reduce objects, or to combine movements in space. By learning to compare models as well as theoretical and experimental results, to optimize situations in a planning and organizational context, and to use logic and deductive reasoning, individuals are able to develop this aptitude as well as the ability to evaluate the factors to be considered in performing the task at hand.

TO ANTICIPATE

in order to …
predict results or behaviours and visualize a finished product before it exists.

TO GENERALIZE

in order to …
reason from the particular to the general, to go from the concrete to the abstract and to promote efficiency.

Through observation and reasoning, individuals identify structures, models and analogies, which they reapply, develop or modify so as to be able to use them in other situations or to anticipate the results of future tasks. The ability to generalize allows individuals to become autonomous in making decisions and in determining what steps to take to attain objectives.

The analysis of numerical sequences, data, algorithmic procedures and relationships among variables as well as the calculation of measurements are mathematical tools that can be used to observe and identify patterns, trends or laws. In addition, mathematics makes it possible to develop proofs. It also involves inductive and deductive reasoning, which are closely linked to the ability to generalize a situation.

TO MAKE DECISIONS

in order to ...
draw conclusions about the solution to a problem, take action concerning a problem or make progress on various issues.

We decide what to do by considering options, setting priorities, evaluating various factors and anticipating the possible consequences of a situation. Decision making is a complex mechanism that helps individuals confirm that they have taken the best course of action so that they can then implement, explain or defend that decision with conviction. The ability to make decisions fosters intellectual independence and leadership skills.

Mathematics provides several tools that can prove useful in the decision-making process. Individuals develop the ability to manipulate and interpret data, to compare them and to identify relationships among them by using various modes of representation, diagrams or models. This involves comparing approaches and seeking an optimal solution that reflects the constraints of a situation. In statistics and probability, the concepts of odds and chance as well as experimentation also come into play in defending the decision taken and assessing the risk or margin of error associated with conjectures, where applicable.

APPENDIX B – EXAMPLES OF STRATEGIES FOR EXERCISING THE COMPETENCIES

It is important to encourage students to develop their autonomy, devise strategies and transfer what they have learned to new situations. This is done by getting them to reflect and ask questions on the concepts and processes involved and the work they have performed. In particular, students use self-evaluation or peer-evaluation grids and benefit from constructive feedback and comments. Cognitive and metacognitive strategies are part and parcel of developing and exercising mathematical competencies; they are integrated into the learning process. Even though these strategies are developed together throughout the cycle, it is possible to emphasize different ones depending on the situation and goal pursued. This ensures that students will master them and call upon them in other situations. Affective and resource management strategies also help students demonstrate their autonomy, fostering the emergence of the positive attitudes needed to see a task through to completion and to derive satisfaction from this.

Learning Strategies	
Affective strategies	Rewarding oneself; speaking to oneself positively; controlling anxiety; keeping one's concentration; finding and maintaining one's motivation; persevering; attributing success to internal factors that can be altered; accepting to take risks; and so on.
Cognitive and metacog-nitive strategies	**Strategies for becoming aware of one's mental activity:** explaining one's line of reasoning; describing one's procedure; recognizing one's weaknesses; identifying what one has learned; defining the conditions for using a procedure and assessing its effectiveness; knowing one's own learning style; and so on. **Planning strategies:** getting an overview of the work to be done; estimating the time required to do it; establishing goals; activating prior knowledge; setting reading goals; extracting explicit or implicit relevant information; defining the task to be performed; analyzing the task; drawing up a work plan; breaking a problem down into more manageable parts; simplifying the problem; giving examples; and so on. **Discrimination strategies:** determining why a given example is not an example of the concept; comparing an example and a counterexample and finding the differences between them; distinguishing between the meaning of terms in everyday language and in mathematical language and assessing one's understanding of these meanings; looking for counterexamples; assessing the relevance of qualitative or quantitative data; and so on. **Organization strategies:** reorganizing a list of words; structuring one's ideas; writing down important ideas; enumerating, grouping, classifying, reorganizing or comparing data; determining the networks of concepts to be used; using lists, diagrams or concrete materials; using different registers of semiotic representation (verbal, symbolic, graphical, tabular); and so on. **Development strategies:** using mnemonic techniques (keywords); creating a mental image; reformulating or rewriting in one's own words (paraphrasing); writing equations, inequalities or systems thereof; summarizing; making analogies; referring to a similar problem that has already been solved; making connections; representing the situation mentally or in writing; proceeding by systematic or guided trials; working backwards; deriving new data from known data; using another viewpoint or strategy (e.g. branch of mathematics, model, process or register); experimenting with different ways of conveying a mathematical message; overcoming an obstacle in one's procedure by assigning an approximate value to a data item; and so on.

Learning Strategies (cont.)	
Cognitive and metacog-nitive strategies (cont.)	**Control strategies:** self-evaluation; giving oneself positive reinforcement; focusing one's attention; assessing the effectiveness of the chosen strategy; reviewing one's work; checking one's solution using examples or a line of reasoning; and so on. **Regulation strategies:** adjusting one's reading speed; rereading to better understand; reviewing steps taken; skipping a step and returning to it later; changing the chosen strategy if necessary; making adjustments; estimating the expected result; assessing whether a new piece of information is consistent with the others; comparing and questioning one's thought processes, procedures and results with those of the teacher or one's peers; and so on. **Generalization strategies:** determining why a given example is an example of the concept; comparing two examples and finding similarities between them; inventing examples; classifying examples according to the concepts; simulating the situation; finding patterns; modelling the situation; developing formulas; interpolating and extrapolating; and so on. **Repetition strategies:** repeating several times; highlighting; underlining; boxing in; making notes selectively; recopying; making lists; and so on. **Strategies for making a process automatic:** finding a model solution and following it step by step; making a list of steps to be followed; carrying out small steps or the entire process; applying conventional or one's own algorithms; comparing one's procedure with that of an expert; and so on.
Resource manage-ment strategies	**Determine available resources:** materials, documents to be consulted; human resources that can be consulted; the times when the teacher or peers can be consulted; and so on. **Manage time effectively:** planning work periods in advance or shorter and more frequent periods; setting sub-objectives to achieve for each work period; and so on. **Manage the study environment:** finding a place to study that is quiet and set up for working; and so on. **Ask for help from others:** asking for help from the teacher or peers; working in small groups; and so on.

Examples of Questions That Students Can Ask to Monitor Their Own Learning

In a learning situation, students can be made to reflect on their actions and to analyze them by asking questions that are consistent with their intentions. Here are a few examples of questions that students could use during or after a learning situation.

Affective strategies	Resource management strategies
– What did I like about this situation? – Am I satisfied with what I have accomplished? – What means did I use to overcome obstacles? – What did I do particularly well in this situation?	– Did I consult reference documents? – Did I accurately estimate the amount of time needed to carry out the activity? – Did I use appropriate methods to maintain my concentration? – Did I consult my teacher at the right times?
Cognitive and Metacognitive Strategies	
Becoming aware of one's mental activity	**Planning**
– What are my strengths and my weaknesses? – What did I learn? How did I learn it? – Can I use this procedure in other situations? – Am I able to explain my line of reasoning? – What aspects of the competencies did I develop?	– Did I define the task to be carried out, estimate the time needed to complete it and identify the relevant information? – Did I use what I already knew about the topic? – Did I need to break the problem down into smaller problems?
Discrimination	**Organization**
– Which terms seem to have a mathematical meaning different from their meaning in everyday language? – Did I need to find a counterexample to refute a conjecture? – Was all the information pertaining to the situation relevant?	– Did I group together, list, classify and compare data, diagrams or networks of concepts and processes? – Did I determine an appropriate network of concepts? – Are the main ideas in my procedure clearly represented?

Cognitive and Metacognitive Strategies (cont.)	
Development	**Control**
– Did I make a note of my comments and questions? – Did I represent the situation mentally or in writing? – Did I refer to a similar problem that has already been solved? – What connections or relationships did I establish? – What data did I derive from the known data?	– What progress does this task help me make? – Did I choose an appropriate strategy? – Can I use a line of reasoning or an example to check my solution?
Regulation	**Generalization**
– Did I use a good reading strategy and take the time I needed to fully understand the description of the situation? – Did I adjust my method to the task? – What explains the difference between the expected result and the actual result? – Which of my classmates' strategies could I also use?	– Did I formulate conjectures (e.g. find reasons for which a given example is an example of the concept)? – Did I compare two examples (similarities) or did I invent examples? – Did I look for explanations as to why a particular action is appropriate? – Are my observations about a particular case applicable to other situations? – Are the statements made or conclusions drawn always true?
Repetition	**Making a process automatic**
– Would I be able to solve the problem again, on my own? – What are the characteristics of a situation that lead me to use the same strategy again? – Am I able to identify and note down the relevant information in order to use it or convey it to someone else?	– Did I practise carrying out a procedure enough times so that I could perform the steps automatically? – Did I make a list of steps to be followed? – Am I able to use previously learned algorithms effectively?

APPENDIX C – LEARNING PROGRESS INDICATORS

A situation or a student's work must be analyzed from two points of view: (1) the specific nature of the task(s) involved in the given situation and (2) overall learning progress as it relates to the competency. The teacher can refer to the following questions and the subsequent tables of learning progress indicators when developing situations or when using evaluation criteria to assess the extent to which a student has developed a particular competency.

General aspects (according to targeted competency)
Solving a situational problem
– Oral or written indication that the student has an appropriate understanding of the situational problem • Is the problem defined and formulated? (a) and (b)[42]
– Mobilization of mathematical knowledge appropriate to the situational problem • Does the approach use the concepts and processes required to solve the situational problem? (b), (d)
– Development of a solution appropriate to the situational problem • Is the problem-solving approach readily evident? (c) • Are all of the required elements used to find the solution? (d) • Is the approach presented in an organized manner? (f)
– Appropriate validation of the steps in the solution • Is it evident that the student validated the solution? (e) • Are the steps in the solution justified? (f)
Using mathematical reasoning
– Formulation of a conjecture appropriate to the situation • Did the conjecture(s) formulated define the situation and meet the intent? (j)
– Correct application of concepts and processes suited to the situation • Does the work submitted show that the student understands and masters the concepts and processes in question? (g), (h), (i)
– Organized implementation of mathematical reasoning suited to the situation • Does the line of reasoning show a structured combination of concepts and processes? (i), (k)
– Proper organization of the steps in an appropriate proof – Correct justification of the steps in a proof • Is the formulated conjecture based on a proof that validates or invalidates that conjecture? (k)

42. The letters in parentheses refer to the learning progress indicators appearing on pages 117 to 119.

General aspects (according to targeted competency) (cont.)
Communication
– Correct illustration of a mathematical concept using another register of semiotic representation • Was the message produced using different registers of representation? (l) • Does the message consist of elements of mathematical language? (p)
– Correct interpretation of a mathematical message involving one or two registers of semiotic representation • Does the message produced indicate that the student interpreted and understood the message received? (n)
– Production of a message in keeping with the terminology, rules and conventions of mathematics, and appropriate to the context • Was the message produced using different registers of representation? (l) • Is the message adapted to the context and the audience? (o) • Does the message consist of elements of mathematical language? (p) • Is the message supported with a structured text? (m), (q) • Does the mathematical message include all of the elements required for its interpretation? (m), (q) • Does the message show that the student used an organized communication plan? (q)

Example of a model for charting learning progress on the cognitive level as it relates to the competency *Solves a situational problem*

Note that for each indicator, the content of each cell includes the content of the preceding cells.

(a) Indicator related to the recognition and understanding of the problem	The student defines the problem **by identifying** information.	The student defines the problem **by identifying and selecting** the relevant information given in the problem.	The student defines the problem **by determining** the implicit information.	The student defines the problem **by analyzing** the information (i.e. by classifying it and identifying the relationships between the items of information).	The student defines the problem **by synthesizing** the information (i.e. by compiling it and presenting it as a coherent whole).
(b) Indicator related to the formulation of the problem	The student formulates the problem **by exploring** the given information in order to highlight the required information and identify any missing information.	The student formulates the problem **by identifying** the concepts and processes and the ways in which they are related.	The student formulates the problem **by translating it** concisely into mathematical language.	The student formulates the problem **by determining** the task(s) to be carried out.	The student formulates the problem **by anticipating** the expected or potential results (nature, category, etc.).
(c) Indicator related to planning a problem-solving approach	The student plans the problem-solving approach **by identifying** constraints and obstacles.	The student plans the problem-solving approach **by determining** the ways or means to overcome obstacles.	The student plans the problem-solving approach **by examining** different possible solutions leading to the product.	The student plans the problem-solving approach **by dividing it up** into sub-problems or **by determining** the essential steps in light of the available resources and the constraints.	The student plans the problem-solving approach **by specifying** the expected results for each step or sub-problem as well as the ways to verify that they have been achieved.
(d) Indicator related to the implementation or application of the problem-solving approach	The student applies the problem-solving approach **by choosing** the required concepts, processes and structures.	The student applies the problem-solving approach **by completing** the operations as planned.	The student applies the problem-solving approach **by analyzing** how the operations are carried out and **determining** whether the resources are appropriate.	The student applies the problem-solving approach **by adjusting** the operations in such a way as to obtain the expected result.	The student applies the problem-solving approach **by taking into account** the fact that each step brings you closer to the expected result.
(e) Indicator related to the validation of the solution	The student validates the solution **by noting** the differences between his/her answer and the expected results and **by examining** the procedure used.	The student validates the solution **by identifying** probable reasons for the difference between his/her answer and expected results or **by revising** the procedure used.	The student validates the solution **by imagining** possible ways to make his/her answer match the expected result or **by finding** a different approach to solving the problem.	The student validates the solution **by choosing** and **applying** methods to obtain the expected results or **by using** a new approach.	The student validates the solution **by reviewing** the results of the operations used or **by reviewing** the steps in the new approach.
(f) Indicator related to the presentation (production) of the solution (approach and final answer)	The student produces an **unstructured, elementary solution** using mathematical language and justifies each of its steps.	The student produces **an elementary solution** using mathematical language and justifies each of its steps.	The student produces **a simple, structured solution** using mathematical language and justifies each of its steps.	The student produces **a complex, structured solution** using mathematical language and justifies each of its steps.	The student produces **a complex, complete and structured solution** using mathematical language and justifies each of its steps.

Example of a model for charting learning progress on the cognitive level as it relates to the competency *Uses mathematical reasoning*

Note that for each indicator, the content of each cell includes the content of the preceding cells.

(g) Indicator related to learning concepts (conceptualization)	The student learns concepts **by determining** their attributes.	The student learns concepts **by recognizing** the differences and similarities between objects or between items in a set of objects (discrimination based on attributes).	The student learns concepts **by dividing** objects into categories (classification).	The student learns concepts **by applying** to a group the attributes observed in a sub-group (generalization).	The student learns concepts **by forming a mental image** based on categorization and by developing a personal definition.
(h) Indicator related to learning processes	The student constructs processes **by becoming aware of** the learning aim and **by identifying** the knowledge needed to construct them.	The student constructs processes **by identifying** differences and similarities or explanatory or logical relationships between prior knowledge and knowledge to be constructed.	The student constructs processes **by organizing them** and by identifying relationships between them (hierarchical, logical, explanatory).	The student constructs processes **by clarifying them** through related learning activities.	The student constructs processes **by using them** in different situations.
(i) Indicator related to the application of concepts and processes	The student applies concepts and processes in a situation involving **pure** application, **in which prior knowledge is not evoked.**	The student applies concepts and processes in a situation involving applications, **in which prior knowledge is evoked and the rule is given.**	The student applies concepts and processes in a situation involving applications, **in which prior knowledge is evoked and the rule is not given.**	**Using a rule or a few simple rules**, the student applies concepts and processes in a **simple** situation involving applications.	**Using a relatively elaborate combination of rules**, the student applies concepts and processes in a **complex** situation involving applications.
(j) Indicator related to the formulation of conjectures	The student formulates a conjecture **by examining** the information provided in the situation.	The student formulates a conjecture **by reorganizing** the information provided in the situation.	The student formulates a conjecture **using examples** in order to put forward a generalization or counterexamples to further define the situation.	The student formulates a conjecture **using similarities** (fairly close resemblances**) or analogies** (overall resemblances) to put forward a generalization.	The student formulates a conjecture **by stating a proposition** that seems to cover most cases and known examples and that would be hard to refute using a counterexample.
(k) Indicator related to the presentation of proofs	The student proves a conjecture **by explaining** what he/she knows about the subject.	The student proves a conjecture **by devising a plan and gathering the resources** (concepts and processes, relations and operators) needed to implement it.	The student proves a conjecture **by determining the structural relationships** between what he/she knows and what he/she must prove.	The student proves a conjecture **by presenting** an approach in which **the sequence of steps and the connections between the steps can be seen.**	The student proves a conjecture **by presenting** an approach in which **each step is relevant and the sequence of steps and the connections between them are clearly stated.**

Example of a model for charting learning progress on the cognitive level as it relates to the competency *Communicates by using mathematical language*

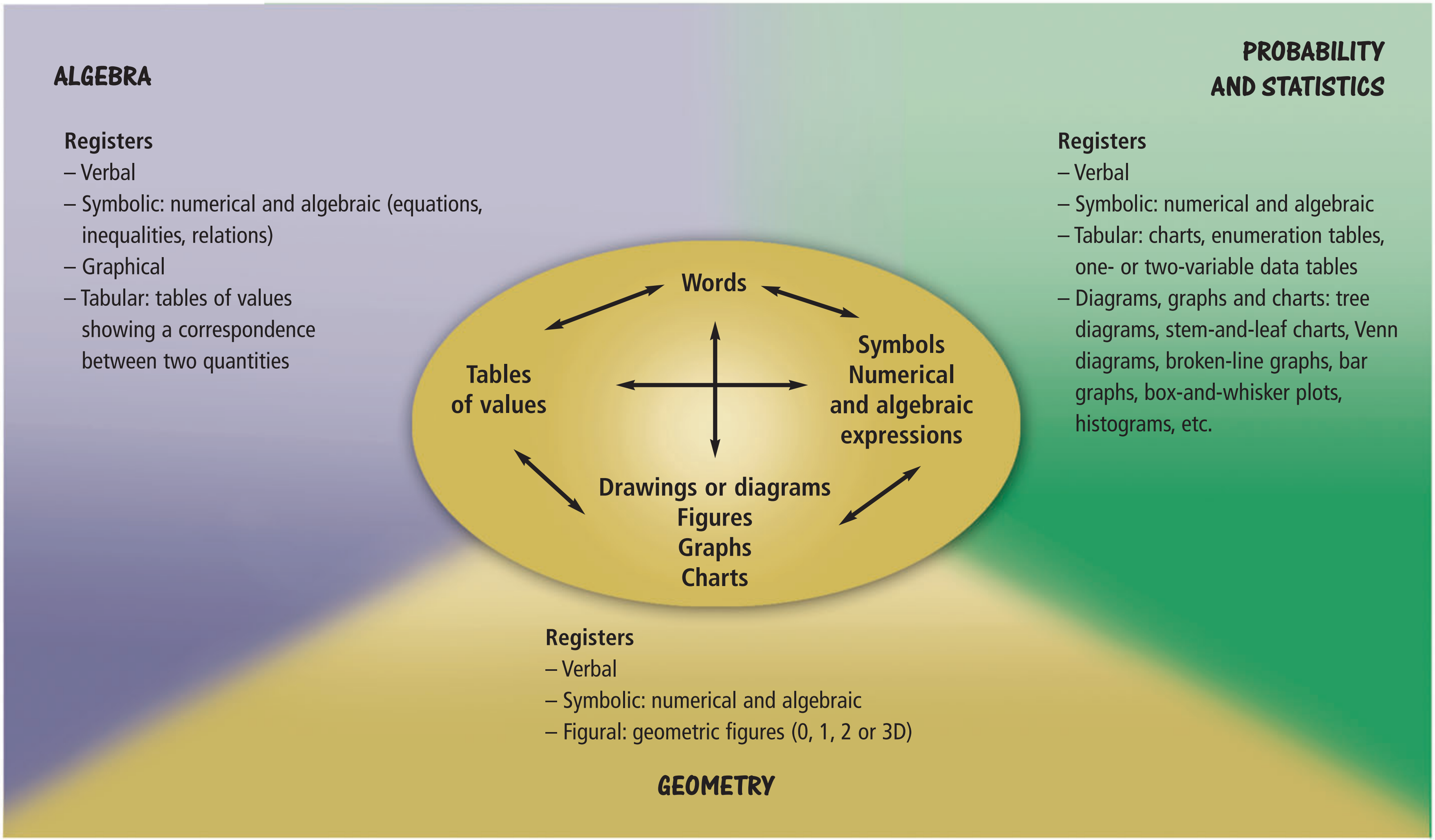
Note that for each indicator, the content of each cell includes the content of the preceding cells.

(l) Indicator related to the registers of semiotic representation	The student translates a mathematical message involving different registers of representation **by exploring** the message.	The student translates a mathematical message involving different registers of representation **by identifying** facts, concepts and relationships.	The student translates a mathematical message involving different registers of representation **by identifying** the relationships between its different parts and **determining** the way it is organized.	The student translates a mathematical message involving different registers of representation **by transcribing** facts, concepts and relationships.	The student translates a mathematical message involving different registers of representation **by organizing** a set of elements and the relationships between these elements, while taking their attributes into account.
(m) Indicator related to the types of sentences or texts used	The student produces an **unstructured, elementary** message (isolated and partially incorrect elements) using mathematical language.	The student produces an **elementary** message (isolated elements) using mathematical language.	The student produces a **simple, structured** message (short or isolated sentences) using mathematical language.	The student produces a **complex, structured** message (text) using mathematical language.	The student produces a **complete, complex and structured** message (text) using mathematical language.
(n) Indicator related to the interpretation of a mathematical message	The student explores a mathematical message **by identifying** data in order to extract certain information.	The student explores a mathematical message **by choosing** data in order to extract certain information.	The student explores a mathematical message **by analyzing** data in order to extract certain information.	The student explores a mathematical message **by summarizing** data in order to extract certain information.	The student explores a mathematical message **by comparing** data in order to explain differences and similarities, and to extract certain information.
(o) Indicator related to the adjustments made in communicating a mathematical message	The student adapts a mathematical message **when told which attitudes, approaches and criteria are to be adjusted**.	The student adapts a mathematical message **by recognizing the attitudes, approaches and criteria to be adjusted**.	The student adapts a mathematical message **by adjusting his/her attitudes, approaches and criteria**.	The student adapts a mathematical message **by recognizing and understanding the attitudes, approaches and criteria to be changed**.	The student adapts a mathematical message **by changing his/her attitudes, approaches and criteria**.
(p) Indicator related to the elements of mathematical language found in the message	The student **uses facts** when he/she produces or interprets a mathematical message.	The student **uses concepts** when he/she produces or interprets a mathematical message.	The student **uses relationships** when he/she produces or interprets a mathematical message.	The student **uses operations** when he/she produces or interprets a mathematical message.	The student **uses structures** when he/she produces or interprets a mathematical message.
(q) Indicator related to the organization of a mathematical message	The student organizes a mathematical message **by determining its purpose (to inform, describe, explain, argue, prove).**	The student organizes a mathematical message **by defining its content** and the expected outcome.	The student organizes a mathematical message **by gathering the necessary information and developing a communication plan.**	The student organizes a mathematical message **by implementing his/her communication plan.**	The student organizes a mathematical message **by adjusting it, if necessary, depending on its purpose and the degree of consistency and accuracy required.**

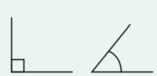
APPENDIX D – COORDINATING ELEMENTS OF MATHEMATICAL LANGUAGE

Coordinating mathematical language involves switching between different registers of semiotic representation in all branches of mathematics.

REGISTERS OF REPRESENTATION, BY BRANCH OF MATHEMATICS



Note: When branches of mathematics are combined, the elements specific to each branch are also combined. For example, the register of geometric figures and the graphical register of algebra are brought together in analytic geometry. When probability and geometry are related, the register of geometric figures and that of numerical expressions involving probability are brought together. The same goes for scatter plots, which may be used to represent experimental data in statistics, algebra and geometry.

Characteristics of the Verbal and Symbolic Registers			
Types of sentences	– Sentences that contain only words: True or false? If a rhombus has four right angles, then the rhombus is a square. – Sentences that contain words and mathematical symbols: What is the solution set for the inequality $2x - 5 \geq 12$? – Sentences that contain only mathematical symbols: $A = \{x \in \mathbb{R} \mid 4 < x \leq 26\} =]4, 26]$		
Meaning of terms	– Terms whose mathematical meaning is the same as their everyday meaning *Length, line, area, etc.* – Terms whose mathematical meaning is different from their everyday meaning *Root, product, radical, function, rational, etc.* – Terms whose mathematical meaning is more specific than their everyday meaning *Similarity, division, average, reflection, relation, etc.* – Terms that have a mathematical meaning only *Hypotenuse, tetrahedron, polyhedron, etc.* – Terms with more than one mathematical meaning *Square, base, degree, inverse, etc.* – Terms with adjectives *Root – Square root* *Polygon – Regular polygon* – Expressions that have a specific meaning in mathematics *If and only if ..., if ... then, and, or*	**Reading symbols and expressions**	– Several words are needed to describe a symbol $\sqrt{}$ square root of … $=$ … is equal to … $\geq$ … is greater than or equal to … – 12 – 5 can be read in different ways twelve minus five from twelve subtract five five less than twelve take away five from twelve the difference between twelve and five five from twelve twelve take away five
Role of symbols	– Symbols used to name objects: $7, \frac{4}{5}, \overline{AB}, \angle$ – Symbols used for operations: $+, -, \times, \div, \sqrt{}, \lvert\ldots\rvert$ – Symbols used in relations: $\subseteq, \in, =, \approx, \cong, \leq, \perp, \sim$ – Symbols for grouping terms (punctuation): $(\ ), [\ ], \{\ \}$ – Graphical symbols 	**Meaning of symbols**	– The order and position of symbols affect their meaning: 46 and 64 2^3 and 3^2 – Different uses affect the meaning: x^{-1} and f^{-1} $4 - 3; -2; \frac{6}{7}; 0,\overline{3}$ ("-" operational, nominative, fraction bar, periodic)

Note: For a more complete list, see *Graphs, Notation and Symbols Used in Secondary School Mathematics,* Ministère de l'Éducation, 1997.

APPENDIX E – AVENUES OF EXPLORATION

This appendix lists situations or figures that students may use to explore, to observe, to deduce measurements and to conjecture (to confirm or refute), as well as statements that may be used in proofs. Not all of these avenues of exploration are prescribed learning content. They can be used to create learning situations and to help students develop and exercise their mathematical competencies.

Avenues of Exploration – First Year of Cycle Two

Arithmetic and Algebra

- Two polynomial functions of degree 0 or 1 are represented by parallel lines if and only if they have the same rate of change.
- When a cylindrical tank is emptied at a constant rate, the relationship between the water level and the volume is proportional and corresponds to a first-degree function.
- The expressions $(x + y)^2$ and $\frac{\left(4x^3 + 8x^2 + 4xy^2\right)}{4x}$ are always equivalent.
- If a fixed amount of time is allotted to carry out work, the relationship between the number of people assigned to perform the tasks and the amount of time each person spends on the work can be represented by a rational function.
- There are as many numbers in the interval [5, 8] as there are in the interval [5, 10].

Geometry

- In similar solids, the ratio of the volumes is equal to the cube of the similarity ratio.
- In similar solids, the ratio of the areas of the corresponding faces is equal to the square of the similarity ratio.
- In a right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the other sides.
- A triangle is right-angled if the square of the length of one of its sides is equal to the sum of the squares of the lengths of the other sides.

Statistics and Probability

- When births are registered in a hospital, the probability of obtaining BGGBGB is greater than that of obtaining BBBBGB. (B: boy, G: girl). (Representativeness: believing that a sequence has more chances of occurring, since it is more representative of the population.)
- There are more ways of forming distinct teams of 3 people than of forming distinct teams of 9 people from a group of 12 people. (Availability: believing that if an event comes more easily to mind, then it has a greater chance of occurring.)
- If two die are rolled simultaneously, obtaining 5 and 6 is as probable as obtaining 6 and 6. (Equiprobability: assigning the same probability value to two events that are not equally probable.)
- Obtaining 2 heads out of 3 when tossing coins is as probable as obtaining 4 heads out of 6 or 20 heads out of 30. (Confusion between probability and proportion)

Avenues of Exploration – *Cultural, Social and Technical* Option

Mathematics is not a deductive science–that's a cliché. When you try to prove a theorem, you don't just list the hypotheses, and then start to reason. What you do is trial and error, experimentation, guesswork.

Paul R. Halmos

This appendix lists examples of families of figures and of situations related to the concepts and processes covered in the *Cultural, Social and Technical* option. They may be explored for the purpose of formulating and validating conjectures. The suggested statements, some of which are false, can be studied, proved or used to create learning situations, deduce certain measurements or justify the steps in a proof or in the solution to a situational problem. These exploration activities involve using proportional reasoning, spatial sense and the properties of geometric transformations. They can be carried out with or without the help of technology. In addition, different voting procedures are described on page 125.

Relations and functions

– All inverses of functions are functions.

– All functions are relations and all relations are functions.

Triangles and right triangles

– The median of a triangle divides the triangle into two triangles with the same area.

– The midpoint of the hypotenuse of a right triangle is equidistant from the three vertices.

– In a right triangle, the length of the side opposite an angle of 30° is equal to half the length of the hypotenuse.

– The lengths of the sides of any triangle ABC are proportional to the sines of the angles opposite these sides: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ (Sine Law).

– The area A of a triangle whose sides measure a, b, and c is: $S = \sqrt{p(p-a)(p-b)(p-c)}$ where p is half the perimeter of the triangle. (Hero's formula)

Congruent triangles

– If the corresponding sides of two triangles are congruent, then the triangles are congruent.

– If two sides and the contained angle of one triangle are congruent to the corresponding two sides and contained angle of another triangle, then the triangles are congruent.

– If two angles and the contained side of one triangle are congruent to the corresponding two angles and contained side of another triangle, then the triangles are congruent.

Similar figures and triangles

– Transversals intersected by parallel lines are divided into segments of proportional lengths.

– The line segment joining the midpoints of two sides of a triangle is parallel to the third side and its length is one-half the length of the third side.

– Any straight line that intersects two sides of a triangle and is parallel to a third side forms a smaller triangle similar to the larger triangle.

– If two angles of one triangle are congruent to the two corresponding angles of another triangle, then the triangles are similar.

– If the lengths of the corresponding sides of two triangles are in proportion, then the triangles are similar.

– If the lengths of two sides of one triangle are proportional to the lengths of the two corresponding sides of another triangle and the contained angles are congruent, then the triangles are similar.

Relative positions of two straight lines in the Cartesian plane

– Two straight lines that are not parallel to the y-axis are parallel if and only if their slopes are equal.

– Two straight lines that are not parallel to the y-axis are perpendicular if and only if their slopes are negative reciprocals.

Distance and midpoint in different situations

– The midpoints of the sides of any quadrilateral are the vertices of a parallelogram.

– The segment joining the midpoints of the nonparallel sides of a trapezoid is parallel to the bases and its length is one-half the sum of the lengths of the bases.

– The segments joining the midpoints of the opposite sides of a quadrilateral and the segment joining the midpoints of the diagonals are concurrent in a point that is the midpoint of each of these segments.

Relations in right triangles where the altitude is drawn from the vertex of the right angle

– The length of a leg of a right triangle is the geometric mean between the length of its projection on the hypotenuse and the length of the hypotenuse.

– The length of the altitude to the hypotenuse of a right triangle is the geometric mean between the lengths of the segments of the hypotenuse.

– The product of the lengths of the legs of a right triangle is equal to the product of the length of the hypotenuse and the length of the altitude to the hypotenuse.

Degrees of a graph and relationships between degrees and edges

– In any simple polyhedron or planar graph, the sum of the number of vertices and the number of faces is equal to the number of edges plus two.

– The sum of the degrees of the vertices of a graph is equal to twice the number of edges in the graph.

– The sum of the degrees of the vertices of a graph is an even number.

– In a graph, there is an even number of vertices whose degrees are odd numbers.

– A connected graph with exactly two vertices whose degrees are odd numbers contains an Euler path.

– A graph contains an Euler circuit if and only if the degrees of all its vertices are even numbers.

– The chromatic number of a graph is less than or equal to $r + 1$, where r is the largest degree of its vertices.

Perimeters and areas of equivalent figures

– Regular polygons have the smallest perimeter of all equivalent polygons with n sides.

– Of two equivalent convex polygons, the polygon with the most sides will have the smaller perimeter. (Ultimately, an equivalent circle will have the smaller perimeter.)

Areas and volumes of equivalent solids

– Cubes have the largest volume of all rectangular prisms with the same total surface area.

– Spheres have the largest volume of all solids with the same total surface area.

– Cubes have the smallest total surface area of all rectangular prisms with the same volume.

– Spheres have the smallest total surface area of all solids with the same volume.

Example

We want to determine which of the three menus is preferred by the 600 students. To do this, we ask each student to indicate his or her preference for each menu. For example, 240 students selected menu C as their first choice, menu A as their second choice and menu B as their third choice. The table on the right shows the number of votes cast for each set of preferences (C-A-B, B-A-C or A-B-C). The outcome according to each voting procedure is indicated below.

Number of students	240	160	200
1st choice	Menu C	Menu B	Menu A
2nd choice	Menu A	Menu A	Menu B
3rd choice	Menu B	Menu C	Menu C

Voting procedures	Definitions	Outcome
Majority rule	The winner is the one with more than half the votes.	In this example, no menu is the winner.
Plurality voting	The winner is the one with the most votes.	In this example, menu C is the winner.
Borda count	Procedure with several candidates in which points are allocated to each candidate according to each voter's preferences (e.g. if there are 4 candidates: 3 points are given to the preferred candidate, 2 points to the candidate ranked second and so on, with 0 points given to the least preferred candidate) (There are variations, for example, if equal rankings are permitted). The "winner" is the one with the most points.	In this example, menu A is the winner.
Condorcet method	In light of the results, the winner of an election would be the one who defeats the other candidates in a one-on-one contest.	In this example, according to the Condorcet method (criterion), menu A is the winner.
Elimination or runoff method	The first step involves counting the first-place votes for each candidate and eliminating the one with the fewest votes. The second step consists in transferring this candidate's first-place votes to the candidate ranked second by these voters and then recounting the first-place votes. If a candidate now has a majority of the votes, this candidate wins the election. Otherwise, the candidate with the fewest votes is eliminated and the procedure is repeated.	In this example, menu B is eliminated in the first round. In the second round, menu C still has 240 votes and menu A, 360 votes (160 + 200). Menu A is the winner.
Approval voting	Procedure that involves voting only once for as many candidates as one wants. The winner is the one who obtains the largest number of votes.	
Proportional representation	Ensures representation that is more or less equivalent to the number of votes obtained. A number of methods exist (e.g. list system, single transferable vote, mixed method).	

Avenues of Exploration – *Technical and Scientific* Option

Mathematical work does not proceed along the narrow logical path of truth to truth to truth, but bravely or gropingly follows deviations through the surrounding marshland of propositions which are neither simply and wholly true nor simply and wholly false.
Seymour Papert

This appendix lists situations, figures and instruments that students may use to explore, to observe, to deduce measurements and to conjecture (to confirm or refute), as well as statements to be used in proofs. Not all of these avenues of exploration are prescribed learning content. They can be used to create learning situations and to help students develop and exercise their mathematical competencies.

Arithmetic and Algebra

- A minimum of two equations is needed to solve a system of first-degree equations in two variables and a minimum of n equations is needed to solve such a system involving n variables.
- All inverses of functions are functions.
- All functions are relations and all relations are functions.
- The optimal solution for a system of inequalities is always found at a vertex of the polygon of constraints.
- In the solution of a system of inequalities, the minimum value corresponds to the lowest vertex and the maximum value corresponds to the highest vertex of the polygon of constraints.
- The relation between the n^{th} term of a sequence and the sum of the first n terms of this sequence corresponds to a polynomial function of degree 2 (Gauss, $\frac{n(n+1)}{2}$).
- Consider two functions: $f(x) = a_1x^2$ and $g(x) = a_2(bx)^2$ where $b > 1$. If $P_1(x_1, y_1)$ is a point belonging to $g(x)$ and $P_0(x_0,y_0)$ is a point belonging to $f(x)$ such that P_1 is the image of P_0 after a transformation of $f(x)$, then the coordinates of P_1 correspond to the coordinates of the point of division located $\frac{1}{b}$ of the way along the segment joining P_0 to the y-axis: $P_1\left(\frac{1}{b}x_0, y_0\right)$. Is this statement true? Can it be true when $b < 1$?

Statistics and Probability

- The calculation of conditional probabilities applies solely to situations in which the events are dependent.
- There is a relationship between the calculation of a weighted mean and the calculation of mathematical expectation.
- Mathematical expectation can be used to determine life expectancy.
- All statistical sampling methods are based on random procedures.
- A sample consisting of only five data values may be representative of a population.
- Weather forecasts are based on subjective probabilities.

Geometry

- If the corresponding sides of two triangles are congruent, then the triangles are congruent.
- If two sides and the contained angle of one triangle are congruent to the corresponding two sides and contained angle of another triangle, then the triangles are congruent.
- If two angles and the contained side of one triangle are congruent to the corresponding two angles and contained side of another triangle, then the triangles are congruent.
- Transversals intersected by parallel lines are divided into segments of proportional lengths.

- The line segment joining the midpoints of two sides of a triangle is parallel to the third side and its length is one-half the length of the third side.
- If two angles of one triangle are congruent to the two corresponding angles of another triangle, then the triangles are similar.
- If the lengths of the corresponding sides of two triangles are in proportion, then the triangles are similar.
- If the lengths of two sides of one triangle are proportional to the lengths of the two corresponding sides of another triangle and the contained angles are congruent, then the triangles are similar.
- In a right triangle, the length of the side opposite an angle of 30° is equal to half the length of the hypotenuse.
- Two straight lines that are not parallel to the *y*-axis are perpendicular if and only if their slopes are negative reciprocals.
- The midpoint of the hypotenuse of a right triangle is equidistant from the three vertices.
- It is possible to obtain an expression that is derived from the sine or cosine ratios pertaining to a right triangle and that is applicable to any triangle *(sine law, cosine law).*
- The length of a leg of a right triangle is the geometric mean between the length of its projection on the hypotenuse and the length of the hypotenuse.
- The length of the altitude to the hypotenuse of a right triangle is the geometric mean between the lengths of the segments of the hypotenuse.
- The product of the lengths of the legs of a right triangle is equal to the product of the length of the hypotenuse and the length of the altitude to the hypotenuse.
- Any diameter perpendicular to a chord divides that chord and each of the arcs that it subtends into two congruent parts.
- The measure of an inscribed angle is one-half the measure of its intercepted arc.
- If a line is perpendicular to a radius of a circle at the endpoint of the radius in the circle, the line is tangent to the circle. The converse is also true.
- In a circle or in congruent circles, two congruent chords are equidistant from the centre and vice versa.
- Two parallel lines, be they secants or tangents, intercept two congruent arcs of a circle.
- If point P is located outside circle O, and if segments PA and PB are tangents to that circle at points A and B respectively, then OP bisects angle APB and $\overline{PA} \cong \overline{PB}$.
- The measure of an angle located between the circumference and the centre of a circle is one-half the sum of the measures of the arcs intercepted by the angle and its vertical angle.
- The measure of an angle located outside a circle is one-half the difference of the measures of the intercepted arcs.
- If two chords of a circle intersect in its interior, the product of the lengths of the segments of one chord is equal to the product of the lengths of the segments of the other chord.
- If secants PAB and PCD of a circle have the same external endpoint P, then $m\,\overline{PA} \cdot m\,\overline{PB} = m\,\overline{PC} \cdot m\,\overline{PD}$.

Locus

- The perpendicular bisector of a segment is a geometric locus.
- Two lines can be constructed parallel to a given segment by drawing the locus of points described by vertex C of a triangle whose base and area are given.
- Given the area of a triangle and the length of one of its sides, the locus corresponding to the possible positions of a vertex of that triangle is a circle.
- Given a circle with centre O and a chord AM, let H be the orthogonal projection of O on chord AM. What is the locus described by point H when M describes the circle?
- The locus of points whose distance from a fixed point is in a constant ratio to its distance from a fixed line is an ellipse, a parabola or a hyperbola, depending on whether the ratio is less than, equal to, or greater than one (Pappus' lemma, 3rd century C.E.).
- The curve representing the locus of points equidistant from a line and a fixed point is of the same shape as the curve representing the graph of a squared proportional relation.
- The geometric locus resulting from the following algorithm is a parabola: 1) Draw two intersecting lines. 2) Graduate these two lines with the same number of graduations. 3) Join the first graduation of one to the last graduation of the other, the second to the penultimate, the third to the ante-penultimate, and so on (construction in the form of Apollonius's envelope of tangents).
- The locus of points where a segment AB can be seen from a given angle corresponds to a circle.

Optimization

- Regular polygons have the smallest perimeter of all equivalent polygons with *n* sides.
- Of two equivalent convex polygons, the polygon with the most sides will have the smaller perimeter. (Ultimately, an equivalent circle will have the smaller perimeter.)
- Cubes have the largest volume of all rectangular prisms with the same total surface area. Cubes have the smallest total surface area of all rectangular prisms with the same volume.
- Spheres have the largest volume of all solids with the same total surface area. Spheres have the smallest total surface area of all solids with the same volume.
- The medians of a triangle determine six equivalent triangles.

Instruments

The study of instruments stemming from the application of mathematical concepts provides opportunities to help students develop intellectually and become aware of the usefulness of mathematics, its widespread use in everyday life and its impact on humankind. It is also an interesting way of introducing students to various trades and occupations.

Pan scale, micrometer caliper, sliding gauge and vernier calipers, gnomon, astrolabe, clock, pendulum, loudspeaker, pulsometer, radar, telescope (Newton's, Mercure's, Galileo's), microscope, overhead projector, binoculars, stroboscope, laser, robot, probe, odometer, multimeter, oscilloscope, pantograph, spinning wheel, arc, set square, carpenter's compass, navigation compass, alidade, altimeter, theodolite, micrometer, goniometer, camera, heliostat, various meteorological instruments (anemometer; Beaufort wind scale), magnetometer, seismograph, GPS (global positioning system), spectrophotometer, compass, galvanometer, combustion analyzer, parabolic antenna, automobile headlights, satellite, medical monitors, electrocardiograph, audiometer, tensiometer, connecting rod, synthesizer, metronome, and so on.

Avenues of Exploration – *Science* Option

It's very easy to avoid becoming intelligent; just follow this simple recipe: fall into the passivity of learned responses and give up trying to ask your own questions.
Albert Jacquard

This appendix lists situations or figures that students may use to explore, to observe, to deduce measurements and to conjecture (to confirm or refute), as well as statements to be used in proofs. Not all of these avenues of exploration are prescribed learning content. They can be used to create learning situations and to help students develop and exercise their mathematical competencies.

- A minimum of n equations is needed to solve a system of first-degree equations in n variables.
- All inverses of functions are functions.
- All functions are relations and all relations are functions.
- The relation between the n^{th} term of a sequence and the sum of the first n terms of this sequence corresponds to a polynomial function of degree 2. (Gauss, $\frac{n(n+1)}{2}$)
- The segment joining the midpoints of two sides of a triangle is parallel to the third side and its length is one-half the length of the third side.
- The midpoint of the hypotenuse of a right triangle is equidistant from the three vertices.
- The segment joining the midpoints of the nonparallel sides of a trapezoid is parallel to the bases and its length is one-half the sum of the lengths of the bases.
- If the corresponding sides of two triangles are congruent, then the triangles are congruent.
- If two sides and the contained angle of one triangle are congruent to the corresponding two sides and contained angle of another triangle, then the triangles are congruent.
- If two angles and the contained side of one triangle are congruent to the corresponding two angles and contained side of another triangle, then the triangles are congruent.
- Plane figures are congruent if and only if all of their corresponding sides and angles are congruent.
- Transversals intersected by parallel lines are divided into segments of proportional lengths.
- If two angles of one triangle are congruent to the two corresponding angles of another triangle, then the triangles are similar.
- If the lengths of the corresponding sides of two triangles are in proportion, then the triangles are similar.
- If the lengths of two sides of one triangle are proportional to the lengths of the two corresponding sides of another triangle and the contained angles are congruent, then the triangles are similar.
- The length of a leg of a right triangle is the geometric mean between the length of its projection on the hypotenuse and the length of the hypotenuse.
- The length of the altitude to the hypotenuse of a right triangle is the geometric mean between the lengths of the segments of the hypotenuse.
- The product of the lengths of the legs of a right triangle is equal to the product of the length of the hypotenuse and the length of the altitude to the hypotenuse.
- The lengths of the sides of any triangle ABC are proportional to the sines of the angles opposite these sides: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ (*sine law*).
- The square of the length of a side of any triangle is equal to the sum of the squares of the lengths of the other two sides minus twice the product of the lengths of the other two sides multiplied by the cosine of the contained angle (*cosine law*).

– The area A of a triangle whose sides measure a, b, and c is: $S = \sqrt{p(p-a)(p-b)(p-c)}$, where p is half the perimeter of the triangle. (Hero's formula)
– The expression $\sin x = \cos\left(\frac{\pi}{2} - x\right)$ is always true.
– For any circle with radius r, the expression $\tan x = \frac{\sin x}{\cos x}$ is true for any real number x.
– Proving trigonometric identities. Examples:
  - $\cos(A + B) = \cos A \cos B - \sin A \sin B$
  - $\cos(90° - (A + B)) = \sin(A + B)$
  - $\text{cosec}\, A(\text{cosec}\, A - \sin A) = \cot^2 A$
  - $\sin\theta = \cos\theta\sqrt{\sec^2\theta - 1}$
  - $2\cos^2\beta - 1 = \frac{\cot\beta - \tan\beta}{\cot\beta + \tan\beta}$
  - $\tan^2\alpha + \cos^2\alpha - 1 = \sin^2\alpha\tan^2\alpha$
  - $\sin^4\alpha - \cos^4\alpha = 2\sin^2\alpha - 1$
  - $\frac{1+\sec\alpha}{\sec\alpha - 1} + \frac{1+\cos\alpha}{\cos\alpha - 1} = 0$
  - $(1+\sec\beta)(\sec\beta - 1) = \frac{\sin\beta\sec\beta}{\cos\beta\,\text{cosec}\,\beta}$
  - $\sin 2\alpha \sec\alpha = 2\sin\alpha$

– Given $\vec{u}$, $\vec{v}$, and $\vec{w}$ which are vectors in the plane, as well as scalars r and s.
  - $\left(r\vec{u} = \vec{o}\right) \Leftrightarrow \left(r = 0 \vee \vec{u} = \vec{o}\right)$
  - If $\vec{u}$ and $\vec{v}$ are non-collinear vectors, then $\left(r\vec{u} = s\vec{v}\right) \Leftrightarrow \left(r = s = 0\right)$.
  - $\left(\vec{w} \text{ are collinear at } \vec{u}\right) \Leftrightarrow \left(\underset{1}{\exists} r \in \mathbb{R} : \vec{w} = r\vec{u}\right)$
  - $\left(\vec{u} \text{ and } \vec{v} \text{ are non-collinear}\right) \Leftrightarrow \left(\forall\vec{w}, \underset{1}{\exists}\in\mathbb{R}, 3, \underset{1}{\exists}\in\mathbb{R} : 3 : \vec{w} = r\vec{u} + s\right.$
  - $\left(\vec{u} \perp \vec{v}\right) \Leftrightarrow \left(\vec{u} \cdot \vec{v} = 0\right)$

– Regular polygons have the smallest perimeter of all equivalent polygons with n sides.
– Of two equivalent convex polygons, the polygon with the most sides will have the smaller perimeter. (Ultimately, an equivalent circle will have the smaller perimeter.)
– Cubes have the largest volume of all rectangular prisms with the same total surface area.
– Spheres have the largest volume of all solids with the same total surface area.
– Cubes have the smallest total surface area of all rectangular prisms with the same volume.
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Bibliography

General Principles

Abrantes, Paulo, Lurdes Serrazina, and Isolina Oliveira. *A Matemática na Educaçïo Básica.* Lisbon: Departamento da Educaçïo Básica, 1999.

Alberta Education. *Alberta Program of Study – M9 Mathematics, Western Canadian Protocol for Collaboration in Basic Education, Grades 6 to 9,* 1996, 192-289.

Alberta Education. *Mathematics: Applied and Pure Programs – 10, 20, 30, Interim,* 1998.

Barberousse, Anouk. "Un dédale conceptuel dans l'empire des probabilités, Dieu joue-t-il aux dés?". *Sciences et avenir.* Paris, hors-série, October-November 2001, no. 128: 16-22.

Burke, Maurice J., and Frances R. Curcio, eds. *Learning Mathematics for a New Century: 2000 Yearbook.* Reston, Virginia: National Council of Teachers of Mathematics, 2000.

COMAP. *For All Practical Purposes – Mathematical Literacy in Today's World,* 5th Edition. New York: W. H. Freeman and Company, 1996.

Crisler, Nancy, Patience Fisher, and Gary Froelich. *Discrete Mathematics Through Applications.* 2nd Edition. New York: W. H. Freeman and Company, 1999.

Cuoco, Albert A., and Frances R. Curcio, eds. *The Roles of Representation in School Mathematics: 2001 Yearbook.* Reston, Virginia: National Council of Teachers of Mathematics, 2001.

Department for Education and Employment of England. *The National Curriculum for England: Mathematics.* [On-line], 1999, <http://www.nc.uk.net>.

Ducrocq, Albert, and André Warusfel. *Les mathématiques, plaisir et nécessité.* Paris: Seuil, 2004. (Collection Points, Sciences)

Guin-Duclosson, Nathalie. "Représentation des connaissances dans l'EIAH AMBRE-add," *Technologies de l'Information et de la Connaissance dans l'Enseignement supérieur et l'industrie,* TICE'2004, Compiègne, October 20-22, 2004, 164-171. [On-line], <archive-edutice.ccsd.cnrs.fr/docs/00/02/75/41/PDF/Duclosson.pdf>.

House, Peggy A., and Arthur F. Coxford, eds. *Connecting Mathematics Across the Curriculum: 1995 Yearbook.* Reston, Virginia: National Council of Teachers of Mathematics, 1995.

Kinney, Margaret J., and Christian R. Hirsch, eds. *Discrete Mathematics Across the Curriculum: 1991 Yearbook.* Reston, Virginia: National Council of Teachers of Mathematics, 1991.

Manitoba. Manitoba Education, Citizenship and Youth. *Applied Mathematics, Consumer Mathematics, A Foundation for Implementation* (equivalent of Secondary III and IV in Québec), 2001.

Mason, John, Leone Burton, and Kaye Stacey. *Thinking Mathematically.* Reading, Massachusetts: Addison-Wesley, 1982.

McGraw, Sue Ann, ed. *Integrated Mathematics – Choices and Challenges.* Reston, Virginia: National Council of Teachers of Mathematics, 2003.

National Council of Teachers of Mathematics. *Principles and Standards for School Mathematics.* Reston, Virginia: National Council of Teachers of Mathematics, 2000.

New Brunswick. Department of Education. *Atlantic Canada Mathematics Program, Grades 7 to 10,* 1999.

Paulos, John Allen. *Innumeracy: Mathematical Illiteracy and Its Consequences.* New York: Vintage Books Edition, 1990.

Société mathématique de France et Société de mathématiques appliquées et industrielles. *L'Explosion des mathematiques.* Paris: SMF and SMAI, 2002.

Steinbring, Heinz, Maria G. Bartolini Bussi, and Anna Sierpinska. *Language and Communication in the Mathematics Classroom.* Reston, Virginia: National Council of Teachers of Mathematics, 1998.

Stiff, Lee V., and Frances R. Curcio, eds. *Developing Mathematical Reasoning in Grades K-12, 1999 Yearbook.* Reston, Virginia: National Council of Teachers of Mathematics, 1999.

Educational Theory

Arsac, Gilbert, et al. *Initiation au raisonnement déductif au collège.* Lyon: Presses Universitaires de Lyon, 1992.

Barbin, Evelyne. "Histoire et enseignement des mathematiques: Pourquoi? Comment?" *Bulletin AMQ,* March 1997, Vol. XXXVII, no. 1: 20-25.

Barbin, Évelyne, Raymond Duval, et al. *Produire et lire des textes de démonstration.* Paris: Ellipses, 2001.

Becker, Jerry P., and Shigeru Shimada, eds. *The Open-Ended Approach, A New Proposal for Teaching Mathematics.* Reston, Virginia: National Council of Teachers of Mathematics, 1997.

Centre de recherche sur l'enseignement des mathématiques. *Les mathematiques de la maternelle jusqu'à 18 ans – Essai d'élaboration d'un cadre global pour l'enseignement des mathematiques.* Nivelles: CREM, 1995.

Centre de recherche sur l'enseignement des mathématiques. *Construire et représenter – Un aspect de la géométrie de la maternelle jusqu'à dix-huit ans.* Nivelles: CREM, 1999. [On-line], <http://www.enseignement.be/@librairie/documents/ressources/081/ constr.asp>.

Centre de recherche sur l'enseignement des mathématiques. *Formes et mouvements – Perspectives pour l'enseignement de la géométrie.* Nivelles: CREM, 1999. [On-line], <http://www.enseignement.be/@librairie/documents/ressources/081/ formes.asp>.

De Serres, Margot, ed. *Intervenir sur les langages en mathématiques et en sciences.* Mont-Royal: Modulo Éditeur, 2003. (Collection Astroïde).

Descaves, Alain. *Comprendre des énoncés, résoudre des problèmes.* Collection Pédagogies pour demain. Didactiques 1er degré. Paris: Hachette Éducation, 1992.

D'Hainault, Louis. *Des fins aux objectifs de l'éducation.* Bruxelles: Labor, 1988.

Duval, Raymond. *Sémiosis et pensée humaine.* Bern: Peter Lang, 1995.

Edwards, Edgar L. Jr. *Algebra for Everyone.* Reston, Virginia: National Council of Teachers of Mathematics, 1990.

Groupement national d'équipes de recherche en didactique des mathématiques. *Algèbre et functions.* Paris: Ministère de l'Éducation Nationale (France), 2000. [On-line], <http://www.eduscol.education.fr/D0124/algebre_et_fonctions.pdf>.

Groupement national d'équipes de recherche en didactique des mathématiques. *L'écrit au collège.* Paris: Ministère de l'Éducation Nationale (France), 1999. [On-line], <http://www.eduscol.education.fr/D0124/greco_ecritcollege.htm>.

Groupement national d'équipes de recherche en didactique des mathématiques. *Preuve et demonstration: quelques questions essentielles.* Paris: Ministère de l'Éducation Nationale (France), 2003. [On-line], <http://www.eduscol.education.fr/D0124/preuve_et_demonstration.pdf>.

Joelants, Nadine, Christian Michaux, and Frédéric Pourbaix. *Mathématiques expérimentales, Activités de modélisation dans l'enseignement des mathématiques au travers des problèmes historiques.* Mons: Le Pentagone, 2001. [On-line], <http://www.enseignement.be/@librairie/documents/ressources/079/index.asp>.

Kilpatrick, Jeremy, Jane Swafford, and Bradford Findell. *Adding It Up, Helping Children Learn Mathematics.* Washington: National Academy Press, 2001.

Richard, Philippe R. *Raisonnement et stratégies de preuve dans l'enseignement des mathematiques.* Bern: Peter Lang, 2004.

Saint-Pierre, Lise. "L'étude et les stratégies d'apprentissage." *Pédagogie collégiale,* vol. 5, no. 2, December 1991 (text taken from "Étudier au collegial: une réalité diversifiée." Proceedings of the 11th annual AQPC conference, June 1991: 109-1 to 109-10).

Van De Walle, John A. *Elementary School Mathematics: Teaching Developmentally.* 4th edition. New York: Longman, 2000.

Wilson, Patricia S., ed. *Research Ideas for the Classroom: High School Mathematics.* New York: Macmillan Library Reference, 1993.

Reference Works

Baruk, Stella. *Dictionnaire de mathématiques élémentaires.* Paris: Seuil, 2000. (Collection Science ouverte).

Bouvier, Alain, and Michel George. *Dictionnaire des mathématiques,* 7th edition. François Le Lyonnais, ed. Paris: PUF, 2005. (Collection Quadrige Dicos Poche).

Burton, David M. *The History of Mathematics: An Introduction.* Dubuque: WCB, 1988.

Droesbeke, Jean-Jacques, and Philippe Tassi. *Histoire de la statistique.* Corrected 2nd edition. Paris: PUF, 1997. (Collection Que sais-je?).

Jacquard, Albert. *Les probabilités.* 6th edition. Paris: PUF, 2000. (Collection Que sais-je?).

Lesmoir-Gordon, Nigel, Will Rood, and Ralph Edney. *Introducing Fractal Geometry.* Lanham, Maryland: Totem Books, 2001.

Mankiewicz, Richard. *The Story of Mathematics.* Princeton, N.J.: Princeton University Press, 2000.

Pallascio, Richard, and Gilbert Labelle. *Mathématiques d'hier et d'aujourd'hui.* Mont-Royal: Modulo, 2000. (Collection Astroïde).

Québec. Ministère de l'Éducation. *Information Document: Graphs, Notation and Symbols Used in Secondary School Mathematics.* (Code:16-3306A). Québec: Gouvernement du Québec, 1997.